SPLIT COMMON FIXED POINT PROBLEMS AND ITS VARIANT FORMS

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ABSTRACT. The split common fixed point problems has found its applications in various branches of mathematics both pure and applied. It provides us a unified structure to study a large number of nonlinear mappings. Our interest here is to apply these mappings and propose some iterative methods for solving the split common fixed point problems and its variant forms, and we prove the convergence results of these algorithms.

As a special case of the split common fixed problems, we consider the split common fixed point equality problems for the class of finite family of quasi-nonexpansive mappings. Furthermore, we consider another problem namely split feasibility and fixed point equality problems and suggest some new iterative methods and prove their convergence results for the class of quasi-nonexpansive mappings.

Finally, as a special case of the split feasibility and fixed point equality problems, we consider the split feasibility and fixed point problems and propose Ishikawa-type extra-gradients algorithms for solving these split feasibility and fixed point problems for the class of quasi-nonexpansive mappings in Hilbert spaces. In the end, we prove the convergence results of the proposed algorithms.

Results proved in this chapter continue to hold for different type of problems, such as; convex feasibility problem, split feasibility problem and multiple-set split feasibility problems.

Keywords Iterative Algorithms, Split Common Fixed Problem, Weak and Strong Convergences.

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1. Introduction

Functional analysis is an abstract branch of mathematics that originated from classical analysis. The impetus came from; linear algebra, problems related to ordinary and partial differential equations, calculus of variations, approximation theory, integral equations, and so on. Functional analysis can be defined as the study of certain topological-algebraic structures and of the methods by which the knowledge of these structures can be applied to analytic problems, see Rudin [1].

Fixed point theory (FPT) is one of the most powerful and fruitful tools of modern mathematics and may be considered a core subject of nonlinear analysis. It has been a nourishing area of research for many mathematicians. The origins of the theory, which date to the later part of the nineteenth century, rest in the use

of successive approximations to establish the existence and uniqueness of the solutions, particularly to differential equations, for example, see [2, 3, 4, 5, 6, 7] and references therein.

The classical importance of fixed point theory in functional analysis is due to its usefulness in the theory of ordinary and partial differential equations. The existence or construction of a solution to a differential equation often reduces to the existence or location of a fixed point for an operator defined on a subset of a space of functions. Fixed point theory had also been used to determine the existence of periodic solutions for functional differential equations when solutions are already known to exist, for example, see [8, 9, 10, 11] and references therein.

Related to the FPT, we have the split common fixed point problems (SCFPP). The SCFPP was introduced and studied by Censor and Segal [12] as a generalization of many existing problems in nonlinear sciences, both pure and applied. Moreover, Censor and Segal [12] had shown that the problem of fixed point, convex feasibility, multiple-set split feasibility, split feasibility and much more can be studied more conveniently as SCFPP. The results and conclusions that are true for the SCFPP continue to hold for these problems, and it shows the significance and range of applicability of the SCFPP. One of the important applications of SCFPP can be seen in intensity modulation radiation therapy (IMRT), for more details, see Censor et al., [13].

This research work falls within the general area of "Nonlinear Functional Analysis", an area with the vast amount of applicability in the recent years, as such becoming the object of an increasing amount of study. We focus on an important topic within this area " **A note on the split common fixed fixed point problem and its variant forms**." In this regard, we discuss the SCFPP and its variant forms. We show that already known problems are special cases of the split common fixed point problems (SCFPP). We use approximation methods to suggest different iterative algorithms for solving SCFPP and its variant forms. In the end, we give the convergence results of these algorithms.

2. Basic Concepts and Definitions

2.1. **Introduction.** In this section, we give some definitions and basic results. We start from the definition of vector space and end with some results from Hilbert spaces. Those results that are commonly used in all the chapters are given in this section, and those results that are relevant to a particular chapter are provided at the beginning of each chapter. In short, this section works as a foundation for the structure of this thesis.

2.2. **Vector Spaces.** Vector spaces play a vital role in many branches of mathematics. In fact, in various practical (and theoretical) problems we have a set V whose elements may be vectors in three-dimensional space, or sequences of numbers, or functions, and these elements can be added and multiplied by constants (numbers) in a natural way, the result being again an element of V. Such concrete situations suggest the concept of a vector space as defined below. The definition will involve a general field $\mathbb{F}$, but in functional analysis, $\mathbb{F}$ will be $\mathbb{R}$ or $\mathbb{C}$. The

elements of $\mathbb{F}$ are called scalars, while in this thesis they will be real or complex numbers.

Definition 2.1. A vector space over a field $\mathbb{F}$ is a nonempty set denoted by V together with addition (+) and scalar multiplication (.) satisfies the following conditions:

(i) x+y=y+x, for all $x, y \in V$;
(ii) x+(y+w)=(x+y)+w, for all $x, y, w \in V$;
(iii) there exists a vector denoted by θ such that $x + \theta = x$, for all $x \in V$;
(iv) for all $x \in V$, there exists a unique vector denoted by (-x) such that $x + (-x) = \theta$;
(v) $\alpha.(\beta.x) = (\alpha.\beta).x$, for all $\alpha, \beta \in \mathbb{F}$ and $x \in V$;
(vi) $\alpha.(x + y) = \alpha.x + \alpha.y$, for all $x, y \in V$ and $\alpha \in \mathbb{F}$;
(vii) $(\alpha + \beta).x = \alpha.x + \beta.x$, for all $\alpha, \beta \in \mathbb{F}$ and $x \in V$;
(viii) there exists $1 \in \mathbb{F}$ such that $1.x = x$, $\forall x \in V$.

Remark 2.2. From now we will drop the dot (.) in the scalar multiplication and denote $\alpha.\beta$ as $\alpha\beta$.

Let $\mathrm{v_1, v_2, v_3, ..., v_n} \in \mathrm{V}$ and $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_\mathrm{n}$ be scalars. Consider the equation:

$$\alpha_1\mathrm{v}_1 + \alpha_2\mathrm{v}_2 + \alpha_3\mathrm{v}_3 + ... + \alpha_\mathrm{n}\mathrm{v}_\mathrm{n} = 0. \tag{2.1}$$

Trivially, $\alpha_1 = \alpha_2 = \alpha_3 = ... = \alpha_n = 0$ solves Equation (2.1). If it is possible to have the solution of Equation (2.1) with at least one of the $\alpha_i' s$ non zero, then the vectors $\mathrm{v_1, v_2, v_3, ..., v_n}$ are called **"Linearly Dependent"** otherwise they are called **"Linearly Independent"**.

If $\mathbb{M} \subseteq$V consist of a linearly independent set of vectors; we say that $\mathbb{M}$ is a linearly independent set.

Definition 2.3. Span of $\mathbb{M}$ (Span$\mathbb{M}$) is defined as the set of all linear combination of $\mathbb{M}$, i.e., Span$\mathbb{M}$ $= \{\alpha_1\mathrm{v}_1 + \alpha_2\mathrm{v}_2 + \alpha_3\mathrm{v}_3 + ..., \mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3, ... \in \mathrm{V}$, where $\alpha_1, \alpha_2, ...$ are scalars$\}$.

Definition 2.4. Let $\mathbb{M} \subseteq$V. $\mathbb{M}$ is said to be basis for the space V, if

(i) $\mathbb{M}$ is a linearly independent set,
(ii) Span$\mathbb{M}$ = V.

Definition 2.5. Let V be a vector space, the dimension of V (dim V) is the number of vectors of the basis of V. V is of finite dimension if its dimension is finite. Otherwise, it is said to be of infinite dimensional space.

Definition 2.6. Let C be a subset of V. C is said to be convex, if for all $x, y \in C$, $\gamma \in [0, 1]$, $(1 - \gamma)x + \gamma y \in C$. In general, for all $x_1, x_2, x_3, ..., x_n \in C$ and for $\gamma_j \geq 0$ such that $\sum_{j=1}^{n} \gamma_j = 1$, the combination $\sum_{j=1}^{n} \gamma_j x_j \in C$ is called the convex combination.

Definition 2.7. A mapping $T : V_1 \to V_2$ is said to be linear, if $\forall u, v \in V_1$ and α, β scalars,

$$T(\alpha u + \beta v) = \alpha T(u) + \beta T(v).$$

Limits (of convergent sequences), differentiation and integration, are examples of a linear map.

Remark 2.8. If in Definition 2.7, the linear space V_2 is replaced by a scalar field $\mathbb{F}$, then the linear map T is called linear functional on V_1.

2.3. **Hilbert Space and its Properties.**

Definition 2.9. Let Y be a linear space. An inner product on Y is a function $\langle ., . \rangle : Y \times Y \to \mathbb{F}$ such that the following conditions are satisfies:

(i) $\langle y, y \rangle \geq 0 \; \forall y \in Y$;
(ii) $\langle y, y \rangle = 0$ if $y = 0$, $\forall y \in Y$;
(iii) $\langle y, z \rangle = \overline{\langle z, y \rangle}$, $\forall y, z \in Y$, where the "bar" indicates the complex conjugation;
(iv) $\langle \alpha x + \beta y, z \rangle = \overline{\alpha} \langle x, z \rangle + \overline{\beta} \langle y, z \rangle$, for all $x, y, z \in Y$ and $\alpha, \beta \in \mathbb{C}$.

Remark 2.10. The pair $(Y, \langle ., . \rangle)$ is called an inner product space. We shall simply write Y for the inner product space $(Y, \langle ., . \rangle)$ when the inner product $\langle ., . \rangle$ is known. Furthermore, if Y is a real vector space, then condition (iii) above reduces to $\langle x, z \rangle = \langle z, x \rangle$ (Symmetry).

Definition 2.11. Let Y be a linear space over $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$). A norm on Y is a real-valued function $\|.\| : Y \to \mathbb{R}$ such that the following conditions are satisfies:

(i) $\|x\| \geq 0$, $\forall x \in Y$;
(ii) $\|x\| = 0$ if $x = 0$, $\forall x \in Y$;
(iii) $\|\alpha x\| = |\alpha| \, \|x\|$, $\forall x \in Y$ and $\alpha \in \mathbb{R}$;
(iv) $\|x + z\| \leq \|x\| + \|z\|$, $\forall x, z \in Y$.

Remark 2.12. A linear space Y with a norm defined on it i.e., $(Y, \|.\|)$ is called a normed linear space. If Y is a normed linear space, the norm $\|.\|$ always induces a metric d on Y given by $d(z, x) = \|z - x\|$ for each $x, z \in Y$, with this, (Y, d) become a metric space. For a quick review of metric space the reader may consult Dunford et al., [16].

Lemma 2.13. Let Y be an inner product space. For arbitrary $x, z \in Y$,

$$|\langle x, z \rangle|^2 \leq \langle x, x \rangle \langle z, z \rangle . \tag{2.2}$$

If x and z are linearly dependent, then Equation (2.2) reduces to

$$|\langle x, z \rangle|^2 = \langle x, x \rangle \langle z, z \rangle .$$

This lemma is known as Cauchy-Schwartz Inequality. For more details about the proof, one is referred to Chidume [14].

Lemma 2.14. A mapping $\|.\| : Y \to \mathbb{R}$ defined by

$$\|x\| = \sqrt{\langle x, x \rangle}, \forall x \in Y$$

is a norm on Y.

Remark 2.15. As the consequence of Lemma 2.14, Equation (2.2) reduces to the following inequality:

$$|\langle x, z \rangle| \leq \|x\| \|z\|, \forall x, z \in Y.$$

Definition 2.16. A sequence $\{y_n\}$ in a normed linear space Y is said to converge to $y \in Y$, if $\forall \epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$, such that $\|y_n - y\| < \epsilon$, $\forall n \geq N_\epsilon$. The vector $y \in Y$ is called the limit of the sequence $\{y_n\}$ and is written as $\lim\limits_{n\to\infty} y_n = y$ or $y_n \to y$, as $n \to \infty$.

Definition 2.17. A sequence $\{y_n\}$ in a normed linear space Y is said to converge weakly to $y \in Y$, if for all $h \in Y^*$ such that $\lim_{n\to\infty} h(y_n) = h(y)$, where Y^* denote the dual space of Y.

Next, we give some results regards to the weak convergence of a sequence. For more details about the proof, see Chidume [14].

Lemma 2.18. Let $\{y_n\} \subseteq E$ (Banach space). Then the following results are satisfies:

(i) $y_n \rightharpoonup y \Leftrightarrow h(y_n) \to h(y)$ for each $h \in E^*$;
(ii) $y_n \to y \Rightarrow y_n \rightharpoonup y$;
(iii) $y_n \rightharpoonup y \Rightarrow \{y_n\}$ is bounded and

$$\|y\| \leq \liminf_{n\to\infty} \|y_n\|;$$

(iv) $y_n \rightharpoonup y$ (in E), $h_n \to h$ (in E^*) $\Rightarrow h_n(y_n) \to h(y)$ (in $\mathbb{R}$).

Remark 2.19. Lemma 2.18 (ii) Shows that strong convergence implies weak convergence. However, the converse may not necessarily be true, that is, in an infinite dimensional space, weak convergence does not always imply strong convergence, while they are the same if the dimension is finite. For the example of weak convergence which is not strong convergence, see Chidume [14] and the references therein.

Definition 2.20. Let C be a subset of H. A sequence $\{y_n\}$ in H is said to be Fejer monotone, if

$$\|y_{n+1} - z\| \leq \|y_n - z\|, \forall n \geq 1, z \in C.$$

Definition 2.21. A sequence $\{y_n\}$ in a normed linear space Y is said to be Cauchy, if $\forall \epsilon > 0$, $\exists N_\epsilon \in \mathbb{N}$ such that $\|y_n - y_m\| < \epsilon$, $\forall n, m \geq N_\epsilon$.

Definition 2.22. A normed linear space Y is said to be complete if and only if every Cauchy sequence in Y converges.

Remark 2.23. With respect to the norm defined in Lemma 2.14, we can define the Cauchy sequence in an inner product space Y. A sequence $\{y_n\}$ in Y is said to be Cauchy if and only if $\langle y_n - y_m, y_n - y_m\rangle^{1/2} := \|y_n - y_m\| \to 0$ as $n, m \to \infty$.

Definition 2.24. An inner product space Y is said to be complete if and only if every Cauchy sequence converges.

Definition 2.25. A complete inner product space is called a "Hilbert Space" and that of normed linear space is known as a "Banach Space".

2.4. **Bounded Linear Map and its Properties.**

Definition 2.26. Let $T : H \to H$ be a linear map. T is said to be bounded, if there exists a constant $M \geq 0$ such that

$$\|T(y)\| \leq M\|y\|, \forall y \in H.$$

Next, we give some results of a linear map that are continuous. For more details about the proof, see Chidume [14].

Lemma 2.27. Let X and Y be normed linear spaces and $T : X \to Y$ be a linear operator. Then the following results are equivalent:

(i) T is continuous;

(ii) T is continuous at the origin i.e., if $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} x_n = 0, \text{ then } \lim_{n\to\infty} Tx_n = 0 \text{ in } Y;$$

(iii) T is Lipschitz, i.e., in the sense that there exists $M \geq 0$ such that

$$\|Tx\| \leq M\|x\|, \forall x \in X;$$

(iv) $T(\Delta)$ is bounded $\Big($in the sense that there exists $M \geq 0$ such that $\|Tx\| \leq M$ for all $x \in \Delta$, where $\Delta := \{x \in X : \|x\| \leq 1\}\Big)$.

Remark 2.28. In the light of Lemma 2.27, we have that a linear map $T : X \to Y$ is continuous iff it is bounded.

Definition 2.29. Let $A : H \to H$ be a bounded linear map. Define a mapping $A^* : H \to H$ by

$$\langle Ay, z\rangle = \langle y, A^*z\rangle, \forall y, z \in H.$$

The mapping A^* is called the adjoint of A.

The following results are fundamental for the adjoint operator on Hilbert space. For the proof, see Chidume [14].

Lemma 2.30. Let $A : H \to H$ be a bounded linear map with its adjoint A^*. Then the following hold:

(i) $(A^*)^* = A$;
(ii) $\|A\| = \|A^*\|$;
(iii) $\|A^*A\| = \|A\|^2$.

2.5. **Some Nonlinear Operators.** Let $T : H \to H$ be a map. A point $x \in H$ is called a **fixed point** of T provided $Tx = x$. We denote the set of fixed point of T by $Fix(T)$, that is

$$Fix(T) = \{x \in H : Tx = x\}.$$

The $Fix(T)$ is closed and convex, for more details, see Goebel and Kirk [15].

T is said to be $\eta-$**strongly monotone**, if there exists a constant $\eta > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \eta \|x - y\|, \forall x, y \in H,$$

and it is said to be **contraction**, if

$$\|Tx - Tz\| \leq k \|x - z\|, \forall x, z \in H, \tag{2.3}$$

where $k \in (0, 1)$.

Remark 2.31. If $T : H \to H$ is a contraction mapping with coefficient $k \in (0, 1)$, then $(I - T)$ is $(1 - k)-$strongly monotone, that is

$$\langle (I - T)w - (I - T)z,\ w - z\rangle \geq (1 - k) \|w - z\|^2, \forall w, z \in H.$$

Proof.

$$\begin{aligned} \langle (I - T)w - (I - T)z,\ w - z\rangle &= \langle\ w - z,\ w - z\rangle + \langle Tz - Tw,\ w - z\rangle \\ &= \langle\ w - z,\ w - z\rangle - \langle Tw - Tz,\ w - z\rangle. \end{aligned} \tag{2.4}$$

On the other hand,

$$\begin{aligned}\langle Tw - Tz,\ w - z\rangle &\le \|Tw - Tz\|\|w - z\| \\ &\le k\|w - z\|^2 \text{ since f is a contraction mapping.} \end{aligned} \tag{2.5}$$

By (2.4) and (2.5), we deduce that

$$\langle (I - T)w - (I - T)z,\ w - z\rangle \ge (1 - k)\,\|w - z\|^2 .$$

And the proof completed. □

Equation (2.3) reduces to the following equation as $k = 1$.

$$\|Tx - Tz\| \le \|x - z\|\,, \forall x, z \in H.$$

This is known as nonexpansive mapping. As a generalization of nonexpansive mapping, we have **asymptotically nonexpansive** (see Goebel and Kirk [17]), this mapping is defined as:

$$\|T^n x - T^n z\| \le k_n \|x - z\|\,, \forall n \ge 1 \text{ and } x, z \in H,$$

where $k_n \subset [1, \infty)$ such that $\lim\limits_{n\to\infty} k_n = 1$.

The map T is said to be **total asymptotically nonexpansive** (see Alber [18]), if

$$\|T^n x - T^n z\|^2 \le \|x - z\|^2 + v_n \eta(\|x - z\|) + \mu_n, \forall n \ge 1 \text{ and } x, z \in H.$$

where $\{v_n\}$ and $\{\mu_n\}$ are sequences in $[0,\infty)$ such that $\lim\limits_{n\to\infty} v_n = 0$, $\lim\limits_{n\to\infty} \mu_n = 0$, and $\eta : \Re^+ \to \Re^+$ is a strictly increasing continuous function with $\eta(0) = 0$. This class of mapping generalizes the class of nonexpansive and asymptotically nonexpansive mappings (for more details see [19, 20] and references therein). And it is said to be $(k, \{\mu_n\}, \{\xi_n\}, \phi)$- total asymptotically strict pseudocontraction, if there exists a constant $k \in [0, 1)$, $\mu_n \subset [0,\infty)$, $\xi_n \subset [0,\infty)$ with $\mu_n \to 0$ and $\xi_n \to 0$ as $n \to \infty$, and continuous strictly increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that

$$\begin{aligned}\|T^n x - T^n y\|^2 \le\ & \|x - y\|^2 + k\,\|(I - T^n)x - (I - T^n)y\|^2 \\ & + \mu_n \phi(\|x - y\|) + \xi_n, \forall x, y \in H.\end{aligned}$$

T is said to be **strictly pseudocontractive** (see Browder and Petryshyn [21]), if

$$\|Tx - Tz\|^2 \le \|x - z\|^2 + k\,\|(I - T)x - (I - T)z\|^2\,, \forall x, z \in H,$$

where $k \in [0, 1)$. And it is said to **pseudocontractive** if

$$\|Tx - Tz\|^2 \le \|x - z\|^2 + \|(I - T)x - (I - T)z\|^2\,, \forall x, z \in H.$$

It is obvious that all nonexpansive mappings and strictly pseudocontractive mappings are pseudocontractive mappings but the converse does not hold.

T is said to be **quasi-nonexpansive** (see Diaz and Metcalf[22]), if $Fix(T) \ne \emptyset$ and

$$\|Tx - z\| \le \|x - z\|,\ \forall x \in H \text{ and } z \in Fix(T).$$

This is equivalent to

$$2\,\langle x - Tx, z - Tx\rangle \le \|Tx - x\|^2\,, \forall x \in H \text{ and } z \in Fix(T). \tag{2.6}$$

Remark 2.32. Every nonexpansive mapping with $Fix(T) \neq \emptyset$ is a quasi-nonexpansive; however, the converse may not necessarily be true. Thus, the class of quasi- nonexpansive mapping generalizes the class of nonexpansive mapping.

The following is an example of a quasi-nonexpansive mapping which is not nonexpansive mapping, for more details, see He and Du[23] and references therein.

Example 2.33. Let $H = \mathbb{R}$, defined $T : Q := [0, \infty) \to \mathbb{R}$ by

$$Ty = \frac{y^2 + 2}{1 + y} \text{ for all } y \in Q.$$

T is said to be $k-$demicontractive, if

$$\|Ty - z\|^2 \leq \|y - z\|^2 + k\|Ty - z\|^2, \forall y \in H \text{ and } z \in Fix(T), \tag{2.7}$$

where $k \in [0, 1)$. Trivially, the class of demicontractive mapping generalizes the class of quasi-nonexpansive mapping for $k \geq 0$.

The following is an example of a demicontractive mapping which is not quasi-nonexpansive mapping, for more details, see Chidume et al., [24] and references therein.

Example 2.34. Define a map $T : l_2 \to l_2$ by

$$T(x_1, x_2, x_3, ...) = -\frac{5}{2}(x_1, x_2, x_3, ...), \text{ for arbitrary vector } (x_1, x_2, x_3, ...) \in l_2.$$

Remark 2.35. If $k = -1$, Equation (2.7) reduces to

$$\|Ty - z\|^2 \leq \|y - z\|^2 - \|Ty - y\|^2, \forall y \in H \text{ and } z \in Fix(T).$$

This is known as **firmly quasi-nonexpansive mapping**. Every strictly pseudocontractive mapping with $Fix(T) \neq \emptyset$ is a demicontractive mapping; however, the converse may not necessarily be true. Thus, the class of demicontractive mapping is more general than the class of strictly pseudocontractive mapping.

The following is an example of demicontractive mapping which is not strictly pseudocontractive mapping, for more details, see Browder and Petryshyn [21] and references therein.

Example 2.36. Let $C = [-1, 1]$ be a sub set of a real Hilbert space H. Define T on C by

$$T(x) = \begin{cases} \frac{2}{3}x\sin(\frac{1}{x}), \text{ if } x \neq 0, \\ 0, \ x = 0. \end{cases}$$

Clearly, 0 is the only fixed point of T . For $x \in C$, we have

$$\begin{aligned} |Tx - 0|^2 &= |Tx|^2 \\ &= \left|\frac{2}{3}x\sin(\frac{1}{x})\right|^2 \\ &\leq \left|\frac{2x}{3}\right|^2 \\ &\leq |x|^2 \\ &\leq |x - 0|^2 + k|Tx - x|^2, \text{ for any } k < 1. \end{aligned}$$

Thus, T is demicontractive mapping. Next, we see that T is not strictly pseudo-contractive mapping. Let $x = \frac{2}{\pi}$ and $z = \frac{2}{3\pi}$, then $|Tx - Tz|^2 = \frac{256}{81\pi^2}$. However,

$$|x - z|^2 + |(I - T)x - (I - T)z|^2 = \frac{160}{81\pi^2}.$$

T is said to be **asymptotically quasi-nonexpansive**, if $Fix(T) \neq \emptyset$ such that for each $n \geq 1$,

$$\|T^n x - z\|^2 \leq t_n \|x - z\|^2, \forall z \in Fix(T) \text{ and } x \in H,$$

where $\{t_n\} \subseteq [1, \infty)$ with $\lim_{n\to\infty} t_n = 1$. It is clear from this definition that every asymptotically nonexpansive mapping with $Fix(T) \neq \emptyset$ is asymptotically quasi-nonexpansive mapping.

Also T is said to be $(\{r_n\}, \{k_n\}, \eta)$**-total quasi-asymptotically nonexpansive mapping,** if

$$\begin{aligned}\|T^n y - z\|^2 \leq \|y - z\|^2 + r_n \eta(\|y - z\|) \\ + k_n, \forall n \geq 1, z \in Fix(T) \text{ and } y \in H,\end{aligned}$$

where $\{r_n\}, \{k_n\}$ are sequences in $[0, \infty)$ such that $\lim_{n\to\infty} r_n = 0$, $\lim_{n\to\infty} k_n = 0$ and $\eta : \Re^+ \to \Re^+$ is a strictly continuous function with $\eta(0) = 0$. This class of mapping, generalizes the class of; quasi-nonexpansive, asymptotically quasi-nonexpansive and total asymptotically nonexpansive mapping.

T is said to be K-Lipschitzian, if

$$\|Ty - Tz\| \leq K \|y - z\|, \forall y, z \in H.$$

It is said to be uniformly K-Lipschitzian, if

$$\|T^n y - T^n z\| \leq K \|y - z\|, \forall y, z \in H.$$

Definition 2.37. A mapping $T : H \to H$ is said to be class$-\tau$ operator, if

$$\langle z - Ty, y - Ty \rangle \leq 0, \forall z \in Fix(T) \text{ and } y \in H.$$

It is important to note that, class$-\tau$ operator is also called directed operator, see Zaknoon [25] and Censo and Segal [12], separating operator, see Cegielski [26] or cutter operator, see Cegielski and Censor [27] and references therein.

Definition 2.38. A self mapping T on H_1 is said to be semi-compact if for any bounded sequence $\{x_n\} \subset H$ with $(I - T)x_n$ converges strongly to 0, there exists a sub-sequence say $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges strongly to x.

Definition 2.39. A self mapping T on C is said to be demiclosed, if for any sequence $\{y_n\}$ in C such that $y_n \rightharpoonup y$ and if the sequence $Ty_n \to z$, then $Ty = z$.

Remark 2.40. In Definition 2.39, if $z = 0$, the zero vector in C, then T is called demiclosed at zero, for more details, see Moudafi [28] and references therein.

Lemma 2.41. (Goebel and Kirk [15]) If a self mapping T on C is a nonexpansive mapping, then T is demiclosed at zero.

Lemma 2.42. (Acedo and Xu [29]) If a self mapping T on C is a $k-$strictly pseudocontractive, then $(T - I)$ is demiclosed at zero.

Lemma 2.43. Let C be a subset of H_1, and P_C be a metric projection from H_1 onto C. Then $\forall y \in C$ and $x \in H_1$,

$$\|x - P_C(x)\|^2 \leq \|y - x\|^2 - \|y - P_C(x)\|^2.$$

For the proof of this lemma, see Li and He [30] and references therein.

Lemma 2.44. For each $x, y \in H_1$, the following results hold.

(i) $\|x+y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2$,
(ii) $\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2$, $\forall\, \alpha \in [0,1]$.

For the proof of this lemma, see Acedo and Xu [29] and references therein.

Lemma 2.45. Let $\{a_n\}$ be a sequence of nonnegative real number such that

$$a_{n+1} \leq (1-\gamma_n)a_n + \sigma_n, n \geq 0,$$

where γ_n is a sequence in $(0,1)$ and σ_n is a sequence of real number such that;

(i) $\lim\limits_{n\to\infty} \gamma_n = 0$ and $\sum \gamma_n = \infty$;
(ii) $\lim\limits_{n\to\infty} \frac{\sigma_n}{\gamma_n} \leq 0$ or $\sum |\sigma_n| < \infty$. Then $\lim\limits_{n\to\infty} a_n = 0$.

For the proof, see Xu [31].

Lemma 2.46. Let $\{x_n\}, \{y_n\}, \{z_n\}$ be sequences of nonnegative real numbers satisfying

$$x_{n+1} \leq (1+z_n)x_n + y_n.$$

If $\sum z_n < \infty$ and $\sum y_n < \infty$, then $\lim\limits_{n\to\infty} x_n$ exist.

For the proof of this lemma, see Tan and Xu [32].

Lemma 2.47. Let $\{x_n\}$ be a Fejer monotone with respect to C, then the following are satisfied:

(i) $x_n \rightharpoonup x^* \in C$ if and only if $\omega_\omega \subset C$;
(ii) $\{P_C x_n\}$ converges strongly to some vector in C;
(iii) if $x_n \rightharpoonup x^* \in C$, then $x^* = \lim\limits_{n\to\infty} P_C x_n$.

For the proof, see Bauschke and Borwein [33].

2.6. **Problem Formulation.** The SCFPP is formulated as follows:

$$\text{Find } x^* \in C := \bigcap_{i=1}^{N} Fix(T_i) \text{ such that } Ax^* \in Q := \bigcap_{j=1}^{M} Fix(G_j). \tag{2.8}$$

In this chapter, we consider $T_i : H_1 \to H_1$, for $i = 1, 2, 3, ..., N$ and $G_j : H_2 \to H_2$, for $j = 1, 2, 3, ..., M$, to be total quasi-asymptotically nonexpansive and or demicontractive mappings.

We denote the solution set of SCFPP (2.8) by

$$\Gamma = \{x^* \in C \text{ such that } Ax^* \in Q\}. \tag{2.9}$$

In sequel, we assume that $\Gamma \neq \emptyset$.

2.7. **Preliminary Results.** A Banach space E satisfies *Opial's condition* (see Opial [83]), if for any sequence $\{x_n\}$ in E such that $x_n \rightharpoonup x,$ as $n \to \infty$ implies that

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|, \ \forall y \in E, y \neq x.$$

T it is said to have *Kadec-Klee property* (see Opial [83]), if for any sequence $\{x_n\}$ in E such that $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|,$ as $n \to \infty$ implies that

$$x_n \to x \text{ and as } n \to \infty.$$

Remark 2.48. Each Hilbert space satisfies the Opial and Kadec-Klee's properties.

The following lemma were taken from Wang et al., [45], we include the proof here for the sake of completeness.

Lemma 2.49. Let $G : H_1 \to H_1$ be a $(\{v_n\}, \{\mu_n\}, \xi)$-total quasi-asymptotically nonexpansive mapping with $Fix(G) \neq \emptyset$. Then, for each $y \in Fix(G)$, $x \in H_1$ and $n \geq 1$, the following inequalities are equivalent:

$$\|G^n x - y\|^2 \leq \|x - y\|^2 + v_n \xi(\|x - y\|) + \mu_n; \tag{2.10}$$

$$2\langle x - G^n x, x - y\rangle \geq \|G^n x - x\|^2 - v_n \xi(\|x - y\|) - \mu_n; \tag{2.11}$$

$$2\langle x - G^n x, y - G^n x\rangle \leq \|G^n x - x\|^2 + v_n \xi(\|x - y\|) + \mu_n. \tag{2.12}$$

Proof. $(i) \Rightarrow (ii)$

$$\begin{aligned}\|G^n x - y\|^2 &= \|G^n x - x + x - y\|^2 \\ &= \|G^n x - x\|^2 + 2\langle G^n x - x, x - y\rangle + \|x - y\|^2,\end{aligned}$$

this imply that

$$\begin{aligned}2\langle G^n x - x, x - y\rangle &= \|G^n x - y\|^2 - \|G^n x - x\|^2 - \|x - y\|^2 \\ &\leq \|x - y\|^2 + v_n \xi(\|x - y\|) + \mu_n - \|G^n x - x\|^2 - \|x - y\|^2.\end{aligned}$$

Thus, we deduce that

$$2\langle x - G^n x, x - y\rangle \geq \|G^n x - x\|^2 - v_n \xi(\|x - y\|) - \mu_n.$$

$(ii) \Rightarrow (iii)$

$$\begin{aligned}\langle x - G^n x, x - y\rangle &= \langle x - G^n x, x - G^n x + G^n x - y\rangle \\ &= \langle x - G^n x, x - G^n x\rangle + \langle x - G^n x, G^n x - y\rangle.\end{aligned}$$

This tends to imply that

$$\begin{aligned}\langle x - G^n x, G^n x - y\rangle &= -\|x - G^n x\|^2 + \langle x - G^n x, x - y\rangle \\ &\geq -\|x - G^n x\|^2 + \frac{1}{2}\|G^n x - x\|^2 \\ &- \frac{1}{2} v_n \xi(\|x - y\|) - \frac{1}{2}\mu_n.\end{aligned}$$

Thus, we deduce that

$$2\langle x - G^n x, y - G^n x\rangle \leq \|x - G^n x\|^2 + v_n \xi(\|x - y\|) + \mu_n.$$

$(iii) \Rightarrow (i)$

$$\begin{aligned}2\langle x - G^n x, y - G^n x\rangle &\leq \|G^n x - x\|^2 + \upsilon_n \xi(\|x-y\|) + \mu_n \\ &= \|G^n x - y\|^2 + 2\langle G^n x - y, y - x\rangle + \|x-y\|^2 \\ &+ \upsilon_n \xi(\|x-y\|) + \mu_n,\end{aligned}$$

thus, we deduce that

$$\|G^n x - y\|^2 \leq \|x-y\|^2 + \upsilon_n \xi(\|x-y\|) + \mu_n.$$

And thus completes the proof. □

Lemma 2.50. *(Mohammed and Kilicman* [86]*)* Let $P_C : H \to C$ be a metric projection such that

$$\langle x_n - x^*, x_n - P_C x_n\rangle \leq 0.$$

Then for each $n \geq 1$,

$$\|P_C x_n - x_n\| \leq \|P_C x_n - x^*\|, \forall x^* \in C.$$

Proof. Let $x^* \in C$, then

$$\begin{aligned}\|x_n - P_C x_n\|^2 &= \|x_n - x^* + x^* - P_C x_n\|^2 \\ &= \|x_n - x^*\|^2 + \|x^* - P_C x_n\|^2 \\ &+ 2\langle x_n - x^*, x^* - P_C x_n\rangle \\ &= \|x_n - x^*\|^2 + \|x^* - P_C x_n\|^2 \\ &+ 2\langle x_n - x^*, x^* - x_n + x_n - P_C x_n\rangle \\ &= \|x_n - x^*\|^2 + \|x^* - P_C x_n\|^2 - 2\|x_n - x^*\|^2 \\ &+ 2\langle x_n - x^*, x_n - P_C x_n\rangle \\ &= \|x^* - P_C x_n\|^2 - \|x_n - x^*\|^2 + 2\langle x_n - x^*, x_n - P_C x_n\rangle \\ &\leq \|x^* - P_C x_n\|^2.\end{aligned}$$

Thus, we conclude that

$$\|x_n - P_C x_n\| \leq \|x^* - P_C x_n\|.$$

□

2.8. **Strong Convergence for the Split Common Fixed Point Problems for Total Quasi-asymptotically Nonexpansive Mappings.**

Theorem 2.51. Let $G : H_1 \to H_1$, $T : H_2 \to H_2$ be $(\{\upsilon_{n_1}\}, \{\mu_{n_1}\}, \xi_1)$, $(\{\upsilon_{n_2}\}, \{\mu_{n_2}\}, \xi_2)$-total quasi-asymptotically nonexpansive mappings and uniformly L_1, L_2-Lipschitzian continuous mappings such that $(G - I)$ and $(T - I)$ are demiclosed at zero. Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint A^*. Also let M and M^* be positive constants such that $\xi(k) \leq \xi(M) + M^* k^2, \forall k \geq 0$. Assume that $\Gamma \neq \emptyset$, and let P_Γ be the metric projection of H_1 onto Γ satisfying

$$\langle x_n - x^*, x_n - P_\Gamma x_n\rangle \leq 0.$$

Define a sequence $\{x_n\}$ by

$$\begin{cases} x_0 \in H_1, \\ u_n = x_n + \gamma A^*(T^n - I)A x_n, \\ x_{n+1} = \alpha_n u_n + (1-\alpha_n) G^n u_n, \forall n \geq 0, \end{cases} \tag{2.13}$$

where γ, L, $\{v_n\}$, $\{\mu_n\}$, $\{\xi_n\}$ and $\{\alpha_n\}$ satisfies the following conditions:

(i) $0 < k < \alpha_n < 1$, $\gamma \in (0, \frac{1}{L^*})$ with $L^* = \|AA^*\|$ and $L = \max\{L_1, L_2\}$;
(ii) $v_n = \max\{v_{n_1}, v_{n_2}\}$, $\mu_n = \max\{\mu_{n_1}, \mu_{n_2}\}$ and $\xi = \max\{\xi_1, \xi_2\}$.

Then $x_n \to x^* \in \Gamma$.

Proof. To show that $x_n \to x^*$, as $n \to \infty$, it suffices to show that

$$x_n \rightharpoonup x^* \text{ and } \|x_n\| \to \|x^*\|, \text{ as } n \to \infty.$$

We divided the proof into five steps as follows:

Step 1. In this step, we show that for each $x^* \in \Gamma$, the following limit exists.

$$\lim_{n\to\infty} \|x_n - x^*\| = \lim_{n\to\infty} \|u_n - x^*\|. \tag{2.14}$$

Now, let $x^* \in \Gamma$. By (2.13) and Lemma 2.49, we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|\alpha_n u_n + (1-\alpha_n)G^n u_n - x^*\|^2 \\
&= \|\alpha_n(u_n - G^n u_n)\|^2 + 2\alpha_n \langle u_n - G^n u_n, G^n u_n - x^*\rangle + \|G^n u_n - x^*\|^2 \\
&= \alpha_n^2 \|u_n - G^n u_n\|^2 + 2\alpha_n \langle u_n - x^* + x^* - G^n u_n, G^n u_n - x^*\rangle \\
&+ \|G^n u_n - x^*\|^2 \\
&= \alpha_n^2 \|u_n - G^n u_n\|^2 + 2\alpha_n \langle u_n - x^*, G^n u_n - x^*\rangle \\
&+ (1-2\alpha_n)\|G^n u_n - x^*\|^2 \\
&= \alpha_n^2 \|u_n - G^n u_n\|^2 + 2\alpha_n \langle u_n - x^*, G^n u_n - u_n + u_n - x^*\rangle \\
&+ (1-2\alpha_n)\|G^n u_n - x^*\|^2 \\
&= \alpha_n^2 \|u_n - G^n u_n\|^2 + 2\alpha_n \langle u_n - x^*, G^n u_n - u_n\rangle \\
&+ 2\alpha_n \langle u_n - x^*, u_n - x^*\rangle + (1-2\alpha_n)\|G^n u_n - x^*\|^2 \\
&\le -\alpha_n(1-\alpha_n)\|u_n - G^n u_n\|^2 + 2\alpha_n\|u_n - x^*\|^2 + \alpha_n v_n \xi(\|u_n - x^*\|) \\
&+ \alpha_n\mu_n + (1-2\alpha_n)\Big(\|u_n - x^*\|^2 + v_n\xi(\|u_n - x^*\|) + \mu_n\Big) \\
&\le -\alpha_n(1-\alpha_n)\|u_n - G^n u_n\|^2 + \|u_n - x^*\|^2 \\
&+ (1-\alpha_n)\Big(v_n\xi(\|u_n - x^*\|) + \mu_n\Big) \\
&= -\alpha_n(1-\alpha_n)\|u_n - G^n u_n\|^2 + \Big(1 + (1-\alpha_n)v_n M^*\Big)\|u_n - x^*\|^2 \\
&+ (1-\alpha_n)\Big(v_n\xi(M) + \mu_n\Big).
\end{aligned} \tag{2.15}$$

On the other hand,

$$\begin{aligned}
\|u_n - x^*\|^2 &= \|x_n - x^* + \gamma A^*(T^n - I)Ax_n\|^2 \\
&= \|x_n - x^*\|^2 + \gamma^2 \|A^*(T^n - I)Ax_n\|^2 \\
&+ 2\gamma \langle x_n - x^*, A^*(T^n - I)Ax_n\rangle,
\end{aligned} \tag{2.16}$$

and

$$\begin{aligned}\gamma^2 \|A^*(T^n - I)Ax_n\|^2 &= \gamma^2 \langle A^*(T^n - I)Ax_n, A^*(T^n - I)Ax_n\rangle \\ &= \gamma^2 \langle AA^*(T^n - I)Ax_n, (T^n - I)Ax_n\rangle \\ &\leq \gamma^2 L^* \|(T^n - I)Ax_n\|^2 . \end{aligned} \tag{2.17}$$

By Lemma 2.49, it follows that

$$\begin{aligned}2\gamma \langle x_n - x^*, A^*(T^n - I)Ax_n\rangle &= 2\gamma \langle Ax_n - T^nAx_n + T^nAx_n - Ax^*, T^nAx_n - Ax_n\rangle \\ &= 2\gamma \langle T^nAx_n - Ax^*, T^nAx_n - Ax_n\rangle \\ &\quad - 2\gamma \|(T^n - I)Ax_n\|^2 \\ &\leq \gamma v_n M^* L^* \|x_n - x^*\|^2 + \gamma(v_n\xi(M) + \mu_n) \\ &\quad - \gamma \|(T^n - I)Ax_n\|^2 . \end{aligned} \tag{2.18}$$

Substituting (2.17) and (2.18) into (2.16), we obtain that

$$\begin{aligned}\|u_n - x^*\|^2 &\leq (1 + \gamma v_n M^* L^*) \|x_n - x^*\|^2 \\ &\quad - \gamma(1 - \gamma L) \|(T^n - I)Ax_n\|^2 + \gamma(v_n\xi(M) + \mu_n). \end{aligned} \tag{2.19}$$

By (2.19) and (2.15), we deduce that

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\leq (1 + (1 - \alpha_n)v_n M^*)\Big((1 + \gamma v_n M^* L^*) \|x_n - x^*\|^2 \\ &\quad - \gamma(1 - \gamma L^*) \|(T^n - I)Ax_n\|^2 + \gamma(v_n\xi(M) + \mu_n)\Big) \\ &\quad - \alpha_n(1 - \alpha_n) \|x_n - G^n u_n\|^2 + (1 - \alpha_n)(v_n\xi(M) + \mu_n) \\ &= \big(1 + (1 - \alpha_n)v_n M^*\big)\big(1 + \gamma v_n M^* L\big) \|x_n - x^*\|^2 \\ &\quad - \gamma(1 - \gamma L^*)(1 + (1 - \alpha_n)v_n M^*) \|(T^n - I)Ax_n\|^2 \\ &\quad - \alpha_n(1 - \alpha_n) \|x_n - G^n u_n\|^2 + (1 + (1 - \alpha_n)v_n M^*)\gamma(v_n\xi(M) + \mu_n) \\ &\quad + (1 - \alpha_n)(v_n\xi(M) + \mu_n) \\ &\leq \big(1 + (1 - \alpha_n)v_n M^*\big)\big(1 + \gamma v_n M^* L\big) \|x_n - x^*\|^2 \\ &\quad - \gamma(1 - \gamma L^*) \|(T^n - I)Ax_n\|^2 - \alpha_n(1 - \alpha_n) \|x_n - G^n u_n\|^2 \\ &\quad + (1 + (1 - \alpha_n)v_n M^*)\gamma(v_n\xi(M) + \mu_n) \\ &\quad + (1 - \alpha_n)(v_n\xi(M) + \mu_n). \end{aligned} \tag{2.20}$$

Thus, we deduce that

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\leq \Big(1 + \gamma v_n M^* L^* + (1 - \alpha_n)v_n M^*(1 + \gamma v_n M^* L^*)\Big) \|x_n - x^*\|^2 \\ &\quad + (1 + (1 - \alpha_n)v_n M^*)\gamma(v_n\xi(M) + \mu_n) + (1 - \alpha_n)(v_n\xi(M) + \mu_n). \end{aligned}$$

This implies that

$$\|x_{n+1} - x^*\|^2 \leq (1 + \beta_n) \|x_n - x^*\|^2 + \eta_n, \tag{2.21}$$

$$\begin{aligned}\text{where } \beta_n &= \gamma v_n M^* L^* + \big(1 - \alpha_n\big)v_n M^*\big(1 + \gamma v_n M^* L^*\big) \\ \eta_n &= \big(1 + (1 - \alpha_n)v_n M^*\big)\gamma\big(v_n\xi(M) + \mu_n\big) + \big(1 - \alpha_n\big)\big(v_n\xi(M) + \mu_n\big). \end{aligned}$$

Clearly, $\sum \beta_n < \infty$ and $\sum \eta_n < \infty$. Moreover, $\beta_n \to 0$ and $\eta_n \to 0$. Hence, by Lemma 2.46, we conclude that $\lim_{n\to\infty} \|x_n - x^*\|$ exist.

We now prove that for each $x^* \in \Gamma$, $\lim_{n\to\infty} \|u_n - x^*\|$ exist.

By (2.20), we deduce that

$$\begin{aligned}\gamma(1-\gamma L^*)\left\|(T^n - I)Ax_n\right\|^2 &\le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\ &\quad + \beta_n \|x_n - x^*\|^2 + \eta_n,\end{aligned} \tag{2.22}$$

and

$$\begin{aligned}\alpha_n(1-\alpha_n)\left\|u_n - G^n u_n\right\|^2 &\le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\ &\quad + \beta_n \|x_n - x^*\|^2 + \eta_n.\end{aligned} \tag{2.23}$$

Thus, as $n \to \infty$, we deduce from (2.22) and (2.23) that

$$\lim_{n\to\infty} \|u_n - G^n u_n\| = 0 \text{ and } \lim_{n\to\infty} \|Ax_n - T^n Ax_n\| = 0. \tag{2.24}$$

Given (2.19), (2.24) and the fact that $\lim_{n\to\infty} \|x_n - x^*\|$ exists, we obtain that

$$\lim_{n\to\infty} \|u_n - x^*\| \text{ exist.}$$

Moreover, by (2.15) and (2.19), we deduce that

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\le \Big(1 + (1-\alpha_n)v_n M^*\Big)\|u_n - x^*\|^2 \\ &\quad + (1-\alpha_n)\Big(v_n \xi(M) + \mu_n\Big),\end{aligned} \tag{2.25}$$

and

$$\begin{aligned}\|u_n - x^*\|^2 &\le (1 + \gamma v_n M^* L^*)\|x_n - x^*\|^2 \\ &\quad + \gamma(v_n \xi(M) + \mu_n).\end{aligned} \tag{2.26}$$

The fact that $\lim_{n\to\infty} \|x_n - x^*\|$ and $\lim_{n\to\infty} \|u_n - x^*\|$ exists, it follows from (2.25) and (2.26) that

$$\lim_{n\to\infty} \|u_n - x^*\| = \lim_{n\to\infty} \|x_n - x^*\|.$$

Step 2. In this step, we show that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0 \text{ and } \lim_{n\to\infty} \|u_{n+1} - u_n\| = 0. \tag{2.27}$$

By (2.13), we have that

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|\alpha_n u_n + (1-\alpha_n)G^n u_n - x_n\| \\ &= \|(1-\alpha_n)(G^n u_n - u_n) + u_n - x_n\| \\ &= \|(1-\alpha_n)(G^n u_n - u_n) + A^*(T^n - I)Ax_n\|.\end{aligned} \tag{2.28}$$

In view of (2.24), we deduce from (2.28) that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{2.29}$$

On the other hand,

$$\begin{aligned}\|u_{n+1}-u_n\| &= \left\|(I+\gamma A^*(T^{n+1}-I)A)x_{n+1}+(I+\gamma A^*(T^n-I)A)x_n\right\|\\ &= \left\|x_{n+1}-x_n+\gamma A^*(T^{n+1}-I)Ax_{n+1}-\gamma A^*(T^n-I)Ax_n\right\|.\end{aligned}$$

Thus, by (2.24) and (2.29) we obtain that

$$\lim_{n\to\infty}\|u_{n+1}-u_n\|=0.$$

Step 3. In this step, we show that

$$\lim_{n\to\infty}\|u_n-Gu_n\|=0 \text{ and } \lim_{n\to\infty}\|Ax_n-Tx_n\|=0. \tag{2.30}$$

The fact that $\lim_{n\to\infty}\|u_n-G^nu_n\|=0$, $\lim_{n\to\infty}\|u_{n+1}-u_n\|=0$ and G is uniformly L-Lipschitzian mapping, we have that

$$\begin{aligned}\|u_n-Gu_n\| &\le \|u_n-G^nu_n\|+\|Gu_n-G^nu_n\|\\ &\le \|u_n-G^nu_n\|+L\left\|u_n-G^{n-1}u_n\right\|\\ &\le \|u_n-G^nu_n\|+L\left\|G^{n-1}u_n-G^{n-1}u_{n-1}\right\|\\ &+L\left\|u_n-G^{n-1}u_{n-1}\right\|\\ &\le \|u_n-G^nu_n\|+L^2\|u_n-u_{n-1}\|\\ &+L\left\|u_n-u_{n-1}+u_{n-1}-G^{n-1}u_{n-1}\right\|\\ &\le \|u_n-G^nu_n\|+L(L+1)\|u_n-u_{n-1}\|\\ &+L\left\|u_{n-1}-G^{n-1}u_{n-1}\right\|.\end{aligned}$$

Thus, as $n\to\infty$, we have that

$$\lim_{n\to\infty}\|u_n-Gu_n\|=0.$$

Similarly, from the fact that, $\lim_{n\to\infty}\|Ax_n-T^nAx_n\|=0$, $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$ and T is uniformly L-Lipschitzian mapping, we deduce that

$$\lim_{n\to\infty}\|Ax_n-TAx_n\|=0.$$

Step 4. In this step, we show that

$$x_n\rightharpoonup x^* \text{ and } u_n\rightharpoonup x^* \text{ as } n\to\infty. \tag{2.31}$$

Since $\{u_n\}$ is bounded, then there exists a sub-sequence $u_{n_i}\subset u_n$ such that

$$u_{n_i}\rightharpoonup x^*, \text{ as } i\to\infty. \tag{2.32}$$

By (2.30) and (2.32), we have that

$$\lim_{i\to\infty}\|u_{n_i}-Gu_{n_i}\|=0. \tag{2.33}$$

From (2.32), (2.33) and the fact that $(G-I)$ is demiclosed at zero, we have that $x^*\in Fix(G)$. By (2.13), (2.32) and the fact $\lim_{n\to\infty}\|Ax_n-T^nAx_n\|=0$, we deduce that

$$x_{n_i}=u_{n_i}-\gamma A^*(T^{n_i}-I)Ax_{n_i}\rightharpoonup x^*.$$

By the definition of A, we get

$$Ax_{n_i}\rightharpoonup Ax^* \text{ as } i\to\infty. \tag{2.34}$$

In view of (2.30), we get

$$\lim_{i\to\infty} \|Ax_{n_i} - TAx_{n_i}\| = 0. \tag{2.35}$$

From (2.34), (2.35) and the fact that $(T-I)$ is demiclosed at zero, we have that $Ax^* \in Fix(T)$. Thus, $x^* \in Fix(G)$ and $Ax^* \in Fix(T)$, and this implies that $x^* \in \Gamma$.

Now, we show that x^* is unique. Suppose to the contrary that there exists another sub-sequence $u_{nj} \subset u_n$ such that $u_{nj} \rightharpoonup y^* \in \Gamma$ with $x^* \neq y^*$. By opial's property of Hilbert space, we have that

$$\begin{aligned}
\liminf_{j\to\infty} \left\|u_{n_j} - x^*\right\| &< \liminf_{j\to\infty} \left\|u_{n_j} - y^*\right\| \\
&= \liminf_{n\to\infty} \|u_n - y^*\| \\
&= \liminf_{j\to\infty} \left\|u_{n_j} - y^*\right\| \\
&< \liminf_{j\to\infty} \left\|u_{n_j} - x^*\right\| \\
&= \liminf_{n\to\infty} \|u_n - x^*\| \\
&= \liminf_{j\to\infty} \left\|u_{n_j} - x^*\right\|.
\end{aligned}$$

Thus, we have

$$\liminf_{j\to\infty} \left\|u_{n_j} - x^*\right\| < \liminf_{j\to\infty} \left\|u_{n_j} - x^*\right\|.$$

This is a contradiction, therefore, $u_n \rightharpoonup x^*$. By using (2.13) and (2.24) , we have

$$x_n = u_n - \gamma A^*(T^n - I)Ax_n \rightharpoonup x^*, \text{ as } n \to \infty.$$

Step 5. In this step, we show that

$$\|x_n\| \to \|x^*\|, \text{ as } n \to \infty. \tag{2.36}$$

To show this, it suffices to show that $\|x_{n+1}\| \to \|x^*\|$ as $n \to \infty$.

By Equation (2.21), Lemma 2.50 and 2.47, and the fact that $\beta_n \to 0$ and $\eta_n \to 0$, we have

$$\begin{aligned}
\Big| \|x_{n+1}\| - \|x^*\| \Big|^2 &\le \|x_{n+1} - x^*\|^2 \\
&\le (1+\beta_n)\|x_n - x^*\|^2 + \eta_n \\
&= \|x_n - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \eta_n \\
&= \|x_n - P_\Gamma x_n + P_\Gamma x_n - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \eta_n \\
&\le 4\|P_\Gamma x_n - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \eta_n.
\end{aligned}$$

Thus, as $n \to \infty$, we have that

$$\begin{aligned}
\lim_{n\to\infty} \Big| \|x_{n+1}\| - \|x^*\| \Big|^2 \le\ & 4\lim_{n\to\infty} \|P_\Gamma x_n - x^*\|^2 \\
& + \lim_{n\to\infty} \beta_n \|x_n - x^*\|^2 + \lim_{n\to\infty} (\eta_n).
\end{aligned}$$

And this implies that

$$\lim_{n\to\infty} \Big| \|x_{n+1}\| - \|x^*\| \Big| = 0.$$

By (2.31) and (2.36), we conclude that $x_n \to x^*$, as $n \to \infty$.

□

2.9. **Strong Convergence for the Split Common Fixed Point Problems for Demicontractive Mappings.** In this section, we considered an algorithm for solving the SCFPP for demicontractive mappings without any prior information on the norm on the bounded linear operator and established the strong convergence results of the proposed algorithm. In the end, we provides some special cases of our suggested methods.

Theorem 2.52. Let $U : H_1 \to H_1$ and $T : H_2 \to H_2$ be k_1, k_2-demicontractive mappings such that $(U - I)$ and $(T - I)$ are demiclosed at zero, $A : H_1 \to H_2$ be a bounded linear operator with its adjoint A^*. Assume that $\Gamma \neq \emptyset$ and let P_Γ be a metric projection from H_1 onto Γ satisfying

$$\langle x_n - x^*, x_n - P_\Gamma x_n \rangle \leq 0.$$

Define $\{x_n\}$ by

$$\begin{cases} x_0 \in H_1 \text{ is arbitrary chosen,} \\ u_n = x_n + \rho_n A^*(T - I)Ax_n, \\ x_{n+1} = (1 - \alpha_n)u_n + \alpha_n U u_n, \forall n \geq 0, \end{cases} \tag{2.37}$$

where $0 < c < \alpha_n < 1 - k$, with $k := \max\{k_1, k_2\}$, and

$$\rho_n = \begin{cases} \frac{(1-k)\|(I-T)Ax_n\|^2}{2\|A^*(I-T)Ax_n\|^2}, TAx_n \neq Ax_n, \\ 0, \qquad\qquad\qquad\qquad \text{otherwise.} \end{cases} \tag{2.38}$$

Then $x_n \to x^* \in \Gamma$.

Proof. To show that $x_n \to x^*$, it suffices to show $x_n \rightharpoonup x^*$ and $\|x_n\| \to \|x^*\|$.

We divided the proof into four steps as follows.

Step 1. In this step, we show that $\{x_n\}$ is a Fejer monotone. This is divided into two cases.

Case 1. If $\rho_n = 0$ and Case 2. If $\rho_n \neq 0$.

Now, let $x^* \in \Gamma$.

Case 1. If $\rho_n = 0$. The fact that U is demicontractive, we have

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|x_n - x^* + \alpha_n(Ux_n - x_n)\|^2 \\ &= \|x_n - x^*\|^2 + 2\alpha_n \langle x_n - x^*, Ux_n - x_n \rangle + \alpha_n^2 \|Ux_n - x_n\|^2 \\ &\leq \|x_n - x^*\|^2 + \alpha_n(k-1) \|Ux_n - x_n\|^2 + \alpha_n^2 \|Ux_n - x_n\|^2 \\ &\leq \|x_n - x^*\|^2 - \alpha_n\big(1 - k - \alpha_n\big) \|Ux_n - x_n\|^2 . \qquad (2.39) \end{aligned}$$

The fact that $0 < \alpha_n < 1 - k$, it follows from (2.39) that $\{x_n\}$ is Fejer monotone.

Case 2. If $\rho_n \neq 0$. Since U and T are demicontractive mappings, we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|u_n - \alpha_n u_n + \alpha_n U u_n - x^*\|^2 \\
&= \|u_n - x^*\|^2 + 2\alpha_n \langle u_n - x^*, U u_n - u_n\rangle + \alpha_n^2 \|U u_n - u_n\|^2 \\
&\leq \|u_n - x^*\|^2 - \alpha_n(1-k) \|U u_n - u_n\|^2 + \alpha_n^2 \|U u_n - u_n\|^2 \\
&\leq \|u_n - x^*\|^2 - \alpha_n\big(1 - k - \alpha_n\big) \|U u_n - u_n\|^2 . \qquad (2.40)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\|u_n - x^*\|^2 &= \|x_n + \rho_n A^*(T-I)Ax_n - x^*\|^2 \\
&= \|x_n - x^*\|^2 + 2\rho_n \langle TAx_n - Ax_n, Ax_n - Ax^*\rangle + \rho_n^2 \|A^*(I-T)Ax_n\|^2 \\
&\leq \|x_n - x^*\|^2 - \rho_n(1-k) \|(T-I)Ax_n\|^2 + \rho_n^2 \|A^*(I-T)Ax_n\|^2 \\
&= \|x_n - x^*\|^2 - \frac{(1-k)^2 \|(I-T)Ax_n\|^2}{2\|A^*(I-T)Ax_n\|^2} \|(T-I)Ax_n\|^2 \\
&+ \frac{(1-k)^2 \|(I-T)Ax_n\|^4}{4\|A^*(I-T)Ax_n\|^4} \|A^*(I-T)Ax_n\|^2 \\
&= \|x_n - x^*\|^2 - \frac{(1-k)^2}{4} \frac{\|(T-I)Ax_n\|^4}{\|A^*(T-I)Ax_n\|^2} . \qquad (2.41)
\end{aligned}$$

Substituting (2.41) into (2.40), we deduce that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \|x_n - x^*\|^2 - \frac{(1-k)^2 \|(T-I)Ax_n\|^4}{4\|A^*(T-I)Ax_n\|^2} \\
&- \alpha_n\big(1 - k - \alpha_n\big) \|U u_n - u_n\|^2 . \qquad (2.42)
\end{aligned}$$

Thus, $\{x_n\}$ is Fejer monotone. Therefore, $\lim_{n\to\infty} \|x_n - x^*\|$ exist.

Step 2. In this step, we show that

$$\lim_{n\to\infty} \|(I-T)Ax_n\| = 0 \text{ and } \lim_{n\to\infty} \|(I-U)x_n\| = 0. \qquad (2.43)$$

Case 1. If $\rho_n = 0$. By (2.38), we see that $\lim_{n\to\infty} \|(I-T)Ax_n\| = 0$. Also by (2.39) and the fact $\lim_{n\to\infty} \|x_n - x^*\|$ exist, it follows that $\lim_{n\to\infty} \|(I-U)x_n\| = 0$.

Case 2. If $\rho_n \neq 0$. The fact $\lim_{n\to\infty} \|x_n - x^*\|$ exist, by (2.42), we deduce that

$$\lim_{n\to\infty} \left(\frac{(1-k)^2 \|(T-I)Ax_n\|^4}{4\|A^*(T-I)Ax_n\|^2} \right) \leq \lim_{n\to\infty} \left(\|x_n - x^*\| - \|x_{n+1} - x^*\|\right) = 0, \qquad (2.44)$$

$$\text{and } \lim_{n\to\infty} \|(I-U)u_n\| \leq \lim_{n\to\infty} \left(\frac{\|x_n - x^*\| - \|x_{n+1} - x^*\|}{c\big(1 - k - \alpha_n\big)} \right) = 0. \qquad (2.45)$$

By (2.44), we have that

$$\lim_{n\to\infty} \left(\frac{\|(T-I)Ax_n\|^2}{\|A^*(T-I)Ax_n\|} \right) = 0. \qquad (2.46)$$

On the other hand,

$$\begin{aligned}\|(T-I)Ax_n\| &= \|A\| \frac{\|TAx_n - Ax_n\|^2}{\|A\| \|TAx_n - Ax_n\|} \\ &\leq \|A\| \frac{\|TAx_n - Ax_n\|^2}{\|A^*(T-I)Ax_n\|}.\end{aligned}$$

Thus, by (2.46) we deduce that

$$\lim_{n\to\infty} \|(T-I)Ax_n\| = 0.$$

Since

$$\begin{aligned}\rho_n \|A^*(T-I)Ax_n\| &= \|u_n - x_n\| \\ &= \frac{(1-k)\|(T-I)Ax\|^2}{2\|A^*(T-I)Ax_n\|}.\end{aligned}$$

Thus, by (2.46) we have

$$\lim_{n\to\infty} \rho_n \|A^*(T-I)Ax_n\| = 0. \tag{2.47}$$

By (2.37), we have that

$$\begin{aligned}\|(U-I)x_n\| &= \|(U-I)u_n - (U-I)\rho_n A^*(T-I)Ax_n\| \\ &\leq \|(U-I)u_n\| + \|(U-I)\rho_n A^*(T-I)Ax_n\|.\end{aligned} \tag{2.48}$$

Given (2.45), (2.47) and the fact that U is bounded. It follows from (2.48) that

$$\lim_{n\to\infty} \|(I-U)x_n\| = 0.$$

Hence, in both case, Equation (2.43) hold.

step 3. In this step, we show that

$$x_n \rightharpoonup x^*, \text{ as } n \to \infty. \tag{2.49}$$

To show this, it suffices to show that $\omega_\omega \subseteq \Gamma$, see Lemma 2.47 (i).

Now, let $q \in \omega_\omega$, this implies that, there exists $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup q$. Since $\lim_{j\to\infty} \left\|Ux_{n_j} - x_{n_j}\right\| = 0$, together with the demiclosed of $(U-I)$ at zero, we conclude that, $q \in Fix(U)$.

On the other hand, since A is bounded, we have that $Ax_{n_j} \rightharpoonup Aq$. By (2.43) and together with the demiclosed of $(T-I)$ at zero, we have that $Aq \in Fix(T)$. Thus, $q \in \Gamma$, this implies that $\omega_\omega \subseteq \Gamma$. Hence, by Lemma 2.47, we conclude that $x_n \rightharpoonup x^*$, as $n \to \infty$.

Step 4. In this step, we show that

$$\|x_n\| \to \|x^*\|, \text{ as } n \to \infty. \tag{2.50}$$

To show (2.50), it suffices to show that $\|x_{n+1}\| \to \|x^*\|$.

By Lemma 2.44 and the fact that $\{x_n\}$ is a Fejer monotone, we have

$$\begin{aligned} \Big| \|x_{n+1}\| - \|x^*\| \Big|^2 &\leq \|x_{n+1} - x^*\|^2 \\ &\leq \|x_n - x^*\|^2 \\ &= \|x_n - P_\Gamma x_n + P_\Gamma x_n - x^*\|^2 \\ &\leq 4\|P_\Gamma x_n - x^*\|^2 . \end{aligned} \tag{2.51}$$

Thus, we deduce that $\|x_{n+1}\| \to \|x^*\|$. By Equation (2.49) and (2.50), we conclude that $x_n \to x^*$ as $n \to \infty$. □

Corollary 2.53. Let $G : H_1 \to H_1$ and $T : H_2 \to H_2$ be (k_{n_1}, k_{n_2})- quasi - asymptotically nonexpansive mappings such that $(G-I)$ and $(T-I)$ are demiclosed at zero, and $A : H_1 \to H_2$ be a bounded linear operator with its adjoint A^*. Also let $L^* = \|AA^*\|$, M and M^* be positive constants such that $\xi(k) \leq \xi(M) + M^*k^2, \forall k \geq 0$. Assume that, $\Gamma \neq \emptyset$, and let P_Γ be a metric projection of H_1 onto Γ satisfying

$$\langle x_n - x^*, x_n - P_\Gamma x_n \rangle \leq 0.$$

Define $\{x_n\}$ by

$$\begin{cases} x_0 \in H_1, \\ u_n = x_n + \gamma A^*(T^n - I)Ax_n, \\ x_{n+1} = \alpha_n u_n + (1-\alpha_n)G^n u_n, \forall n \geq 0, \end{cases} \tag{2.52}$$

where $\alpha_n \subset (0,1)$, $\gamma \in (0, \frac{1}{L^*})$, $L = \max\{L_1, L_2\}$ and $k_n = \max\{k_{n_1}, k_{n_2}\}$. Then $x_n \to x^* \in \Gamma$.

Proof. G and T are $(\{v_n\}, \{\mu_n\}, \xi)-$ total quasi-asymptotically nonexpansive mappings with $\{v_n\} = \{k_n - 1\}, \mu_n = 0$ and $\xi(k) = k^2, \forall k \geq 0$. Moreover, G and T are uniformly $k_{n_1}, k_{n_2}-$ Lipschitzian mappings. Therefore, all the conditions in Theorem 2.13 are satisfied. Hence, the conclusion of this corollary follows directly from Theorem 2.13. □

Corollary 2.54. Let $G : H_1 \to H_1$ and $T : H_2 \to H_2$ be two quasi-nonexpansive mappings such that $(G-I)$ and $(T-I)$ are demiclosed at zero. And let A be a bounded linear operator with its adjoint A^*. Assume that, $\Gamma \neq \emptyset$, and let P_Γ be a metric projection of H onto Γ satisfying

$$\langle x_n - x^*, x_n - P_\Gamma x_n \rangle \leq 0.$$

Define $\{x_n\}$ by

$$\begin{cases} x_0 \in H_1; \\ u_n = x_n + \gamma A^*(T - I)Ax_n; \\ x_{n+1} = \alpha_n u_n + (1-\alpha_n)Gu_n, \forall n \geq 0, \end{cases} \tag{2.53}$$

where $\{\alpha_n\} \subset (0,1)$ and $\gamma \in (0, \frac{1}{L^*})$ with $L^* = \|AA^*\|$. Then $x_n \to x^* \in \Gamma$.

Proof. G and T are $(1)-$quasi-asymptotically nonexpansive mappings. Moreover, G and T are uniformly $1-$ Lipschitzian mappings. Therefore, all the conditions of Corollary 2.53 are satisfied. Hence, the conclusions of this corollary follow directly from Corollary 2.53.

□

Corollary 2.55. Let H_1, H_2, A, A^*, P_Γ and $\{x_n\}$ be as in Theorem 2.52. Also let $U : H_1 \to H_1$ and $T : H_2 \to H_2$ be quasi nonexpansive mappings such that $(U - I)$ and $(T - I)$ are demiclosed at zero. Assume that $\Gamma \neq \emptyset$. Then $x_n \to x^* \in \Gamma$.

Proof. Since T is quasi-nonexpansive, clearly T is 0-demicontractive. Hence, all the hypothesis of Theorem 2.52 are satisfied. Therefore, the proof of this corollary follows trivially from Theorem 2.52. □

Corollary 2.56. Let H_1, H_2, A, A^*, P_Γ and $\{x_n\}$ be as in Theorem 2.52. Also let $U : H_1 \to H_1$ and $T : H_2 \to H_2$ be firmly quasi nonexpansive mappings such that $(U - I)$ and $(T - I)$ are demiclosed at zero. And assume that $\Gamma \neq \emptyset$. Then $x_n \to x^* \in \Gamma$.

Corollary 2.57. Let H_1, H_2, A, A^*, P_Γ and $\{x_n\}$ be as in Theorem 2.52. Also let $U : H_1 \to H_1$ and $T : H_2 \to H_2$ be directed operators such that $(U - I)$ and $(T - I)$ are demiclosed at zero and assume that $\Gamma \neq \emptyset$. Then, $x_n \to x^* \in \Gamma$.

Proof. Since T is directed operator, clearly T is $(-1)-$demicontractive. Hence, all the hypothesis of Theorem 2.52 are satisfied. Therefore, the proof of this corollary follows trivially from Theorem 2.52.

□

2.10. **Application to Variational Inequality Problems.** Let $T : C \to H_1$ be a nonlinear mapping. The variational inequality problem with respect to C consist as finding a vector $x^* \in C$ such that

$$\langle Tx^*, x - x^* \rangle \geq 0,\ \forall\ x \in C. \tag{2.54}$$

We denote the solution set of Variational Inequality Problem (2.54) by $VI(T, C)$.

It is easy to see that

$$\text{find } x^* \in VI(T, C) \text{ if and only if } x^* \in Fix(P_C(I - \beta T)), \tag{2.55}$$

where P_C is the metric projection from H_1 onto C and β is a positive constant.

Let $Q := Fix(P_C(I - \beta T))$ (the fixed point set of $P_C(I - \beta T)$) and $A = I$ (the identity operator on H_1), then Equation (2.54) can be written as;

$$\text{find } x^* \in C \text{ such that } Ax^* \in Q. \tag{2.56}$$

2.11. **On Synchronal Algorithms for Fixed and Variational Inequality Problems in Hilbert Spaces.** The aim of this section is to expand the general approximation method proposed by Tian and Di [89] to the class of $(k, \{\mu_n\}, \{\xi_n\}, \phi)$-total asymptotically strict pseudocontraction and uniformly M-Lipschitzian mappings to solve the fixed point problem as well as variational inequality problem in the frame work of Hilbert space. The results presented in this paper extend, improve and generalize several known results in the literature.

2.12. **Preliminaries.** In the sequel we shall make use of the following lemmas in proving the main results of this section.

Lemma 2.58. [90]Let H be a Hilbert space, there hold the following identities;

(i) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle, \forall x, y \in H$;
(ii) $\|tx + (1 - t)y\|^2 = t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)\|x - y\|^2, \forall t \in [0, 1]$ and $x, y \in H$;

(iii) if $\{x_n\}$ is a sequence in H such that $x_n \rightharpoonup z$, then

$$\limsup_{n\to\infty} \|x_n - y\|^2 = \limsup_{n\to\infty} \|x_n - z\|^2 + \limsup_{n\to\infty} \|z - y\|^2, \forall y \in H.$$

Lemma 2.59. [91]Let C be a nonempty closed convex subset of a real Hilbert space H and let $T : C \to C$ be a $(k,\ \{\mu_n\},\ \{\xi_n\},\ \phi)$- total asymptotically strict pseudocontraction mapping and uniformly L-Lipschitzian. Then $I-T$ is demiclosed at zero in the sense that if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x^*$, and $\limsup_{n\to\infty} \|(T^n - I)x_n\| = 0$, then $(T - I)x^* = 0$.

Lemma 2.60. [89] *Assume that* $\{a_n\}$ *is a sequence of nonnegative real number such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \sigma_n, n \geq 0,$$

where γ_n *is a sequence in* $(0, 1)$ *and* σ_n *is a sequence of real number such that;*

(i) $\lim_{n\to\infty} \gamma_n = 0$ and $\sum \gamma_n = \infty$*;*
(ii) $\lim_{n\to\infty} \frac{\sigma_n}{\gamma_n} \leq 0$ or $\sum |\sigma_n| < \infty$. Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.61. [89]Let $F : H \to H$ be a η- strongly monotone and L-Lipschitzian operator with $L > 0$ and $\eta > 0$. Assume that $0 < \mu < \dfrac{2\eta}{L^2}$, $\tau = \mu\left(\eta - \dfrac{L^2\mu}{2}\right)$ and $0 < t < 1$. Then

$$\|(I - \mu tF)x - (I - \mu tF)y\| \leq (1 - \tau t)\|x - y\|.$$

Lemma 2.62. (Bulama and Kilicman [84]) Let $S : C \to H$ be a uniformly L-Lipschitzian mapping with $L \in (0, 1]$. Define $T : C \to H$ by $T^{\beta_n}x = \beta_n x + (1 - \beta_n)S^n x$ with $\beta_n \in (0, 1)$ and $\forall x \in C$. Then T^{β_n} is nonexpansive and $Fix(T^{\beta_n}) = Fix(S^n)$.

Proof. Let $x, y \in C$, from lemma (2.1(ii)), we have

$$\begin{aligned}
\left\|T^{\beta_n}x - T^{\beta_n}y\right\|^2 &= \|\beta_n(x - y) + (1 - \beta_n)(S^n x - S^n y)\|^2 \\
&= \beta_n \|x - y\|^2 + (1 - \beta_n)\|S^n x - S^n y\|^2 \\
&- \beta_n(1 - \beta_n)\|(x - y) - (S^n x - S^n y)\|^2 \\
&\leq \beta_n \|x - y\|^2 + (1 - \beta_n)\|S^n x - S^n y\|^2 \\
&\leq (L^2 + \beta_n(1 - L^2))\|x - y\|^2,
\end{aligned}$$

since $L \in (0, 1]$ and $\beta_n \in (0, 1)$, it follow that, T^{β_n} is nonexpansive, and it is not difficult to see that $Fix(T^{\beta_n}) = Fix(S^n)$. □

Lemma 2.63. [88]Let H be a real Hilbert space, $f : H \to H$ be a contraction with coefficient $0 < \alpha < 1$ and $F : H \to H$ be a L-Lipschitzian continuous operator and η-strongly monotone operator with $L > 0$ and $\eta > 0$. Then for $0 < \gamma < \frac{\mu\eta}{\alpha}$,

$$\langle x - y, (\mu F - \gamma f)x - (\mu F - \gamma f)y\rangle \geq (\mu\eta - \gamma\alpha)\|x - y\|^2.$$

Theorem 2.64. Let $T : H \to H$ be a $(k, \{\mu_n\}, \{\xi_n\}, \phi)$- total asymptotically strict pseudocontraction mapping and uniformly M-Lipschitzian with $\phi(t) = t^2, \forall t \geq 0$ and $M \in (0, 1]$. Assume that $Fix(T^n) \neq \emptyset$, and let f be a contraction with coefficient $\beta \in (0, 1)$, $G : H \to H$ be a η-strongly monotone and L-Lipschitzian operator

with $L > 0$ and $\eta > 0$ respectively. Assume that $0 < \gamma < \mu(\eta - \frac{\mu L^2}{2})/\beta = \frac{\tau}{\beta}$ and let $x_0 \in H$ be chosen arbitrarily, $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in (0,1) satisfying the following conditions:

$$\begin{cases} \text{(i)} \lim\limits_{n\to\infty} \alpha_n = 0 \text{ and } \sum \alpha_n = \infty; \\ \text{(ii)} \sum |\alpha_{n+1} - \alpha_n| < \infty, \sum |\beta_{n+1} - \beta_n| < \infty \text{ and } \sum |\beta_n| < \infty; \\ \text{(iii)} 0 \leq k \leq \beta_n < a < 1, \forall n \geq 0. \end{cases} \tag{2.57}$$

Let $\{x_n\}$ be a sequence defined by

$$\begin{cases} T^{\beta_n} = \beta_n I + (1-\beta_n)T^n; \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n \mu G)T^{\beta_n} x_n, \end{cases} \tag{2.58}$$

then $\{x_n\}$ converges strongly to a common fixed of T^n which solve the variational inequality problem

$$\langle (\gamma f - \mu G)x^*, x - x^* \rangle \leq 0, \forall x \in Fix(T^n). \tag{2.59}$$

Proof. The proof is divided into five steps as follows.

Step 1. In this step, we show that

$$T^{\beta_n} \text{ is nonexpansive and } Fix(T^{\beta_n}) = Fix(T^n). \tag{2.60}$$

The proof follows directly from lemma (2.62).

Step 2. In this step, we show that

$$\{x_n\}, \{T^n x_n\}, \{f(x_n)\} \text{ and } \{GT^n x_n\} \text{ are all bounded.} \tag{2.61}$$

Let $x^* \in Fix(T^n)$, from (2.58) and lemma (2.61), and the fact that f is a contraction, we have

$$\begin{aligned} \|x_{n+1} - x^*\| &= \left\|\alpha_n \gamma f(x_n) + (I - \alpha_n \mu G)T^{\beta_n} x_n - x^*\right\| \\ &= \left\|\alpha_n(\gamma f(x_n) - \mu G x^*) + (I - \alpha_n \mu G)T^{\beta_n} x_n - (I - \alpha_n \mu G)x^*\right\| \\ &\leq (1 - \alpha_n \tau)\|x_n - x^*\| + \alpha_n \|\gamma(f(x_n) - f(x^*)) + \gamma f(x^*) - \mu G x^*)\| \\ &\leq (1 - \alpha_n(\tau - \gamma\beta))\|x_n - x^*\| + \alpha_n \|\gamma f(x^*) - \mu G x^*)\| \\ &\leq \max\left\{\|x_n - x^*\|, \frac{\|\gamma f(x^*) - \mu G x^*)\|}{(\tau - \gamma\beta)}\right\}. \end{aligned}$$

By using induction, we have

$$\|x_{n+1} - x^*\| \leq \max\left\{\|x_0 - x^*\|, \frac{\|\gamma f(x^*) - \mu G x^*)\|}{(\tau - \gamma\beta)}\right\}. \tag{2.62}$$

Hence $\{x_n\}$ is bounded, and also

$$\begin{aligned} \|T^n x_n - x^*\|^2 &\leq \|x_n - x^*\|^2 + k\|x_n - x^* - (T^n x_n - x^*)\|^2 + \mu_n \phi(\|x_n - x^*\|) + \xi_n \\ &= \|x_n - x^*\|^2 + k\|x_n - x^*\|^2 + k\|T^n x_n - x^*\|^2 \\ &+ 2k\|x_n - x^*\|\|T^n x_n - x^*\| + \mu_n \|x_n - x^*\|^2 + \xi_n \\ &\leq (1 + k + \mu_n)\|x_n - x^*\|^2 + 2k\|x_n - x^*\|\|T^n x_n - x^*\| \\ &+ k\|T^n x_n - x^*\|^2 + \xi_n. \end{aligned} \tag{2.63}$$

From (2.63), we deduce that

$$(1-k)\left\|T^n x_n - x^*\right\|^2 - 2k\left\|x_n - x^*\right\|\left\|T^n x_n - x^*\right\| \\ -(1+k+\mu_n)\left\|x_n - x^*\right\|^2 - \xi_n \leq 0.$$

This implies that

$$\begin{aligned}\left\|T^n x_n - x^*\right\| &\leq \frac{k\left\|x_n - x^*\right\|}{(1-k)} \\ &+ \frac{\sqrt{4k^2\left\|x_n - x^*\right\|^2 + 4(1-k)\{(1+k+\mu_n)\left\|x_n - x^*\right\|^2 + \xi_n\}}}{2(1-k)} \\ &= \frac{k\left\|x_n - x^*\right\| + \sqrt{(1+(1-k)\mu_n)\left\|x_n - x^*\right\|^2 + (1-k)\xi_n}}{(1-k)} \\ &\leq \frac{k\left\|x_n - x^*\right\| + (1+(1-k)\mu_n)\left\|x_n - x^*\right\|^2 + (1-k)\xi_n}{(1-k)}\end{aligned}$$

$$\left\|T^n x_n - x^*\right\| \leq M^*, \tag{2.64}$$

where M^* is chosen arbitrarily such that

$$\sup\left(\frac{k\left\|x_n - x^*\right\| + (1+(1-k)\mu_n))\left\|x_n - x^*\right\|^2 + (1-k)\xi_n}{(1-k)}\right) \leq M^*.$$

It follows from (2.64) that $\{T^n x_n\}$ is bounded. Since G is L-Lipschitzian, f is contraction and the fact that $\{x_n\}, \{T^n x_n\}$ are bounded, it is easy to see that $\{GT^n x_n\}$ and $\{f(x_n)\}$ are also bounded.

Step 3. In this step, we show that

$$\lim_{n\to\infty}\left\|x_{n+1} - x_n\right\| = 0. \tag{2.65}$$

Now,

$$\begin{aligned}\left\|x_{n+2} - x_{n+1}\right\| &= \Big(\alpha_{n+1}\gamma f(x_{n+1}) + (I - \alpha_{n+1}\mu G)T^{\beta_{n+1}}x_{n+1}\Big) \\ &- \Big(\alpha_n\gamma f(x_n) + (I - \alpha_n\mu G)T^{\beta_n}x_n\Big) \\ &= \alpha_{n+1}\gamma(f(x_{n+1}) - f(x_n)) + (\alpha_{n+1} - \alpha_n)\gamma f(x_n) \\ &+ (I - \alpha_{n+1}\mu G)T^{\beta_{n+1}}x_{n+1} - (I - \alpha_{n+1}\mu G)T^{\beta_n}x_n \\ &+ (\alpha_n - \alpha_{n+1})\mu GT^{\beta_n}x_n,\end{aligned}$$

this turn to implies that

$$\begin{aligned}\left\|x_{n+2} - x_{n+1}\right\| &\leq \alpha_{n+1}\gamma\beta\left\|x_{n+1} - x_n\right\| + (1 - \alpha_{n+1}\tau)\left\|T^{\beta_{n+1}}x_{n+1} - T^{\beta_n}x_n\right\| \\ &+ \left|\alpha_{n+1} - \alpha_n\right|\Big(\gamma\left\|f(x_n)\right\| + \mu\left\|GT^{\beta_n}x_n\right\|\Big) \\ &\leq \alpha_{n+1}\gamma\beta\left\|x_{n+1} - x_n\right\| + (1 - \alpha_{n+1}\tau)\left\|T^{\beta_{n+1}}x_{n+1} - T^{\beta_n}x_n\right\| \\ &+ \left|\alpha_{n+1} - \alpha_n\right| N_1,\end{aligned} \tag{2.66}$$

where N_1 is chosen arbitrarily so that $\sup_{n\geq 1}\Big(\gamma\left\|f(x_n)\right\| + \mu\left\|GT^{\beta_n}x_n\right\|\Big) \leq N_1$.

On the other hand,

$$\begin{aligned}\left\|T^{\beta_{n+1}}x_{n+1}-T^{\beta_n}x_n\right\| &\leq \left\|T^{\beta_{n+1}}x_{n+1}-T^{\beta_{n+1}}x_n\right\|+\left\|T^{\beta_{n+1}}x_n-T^{\beta_n}x_n\right\| \\ &\leq \|x_{n+1}-x_n\|+|\beta_{n+1}-\beta_n|\,\|x_n\|+|\beta_{n+1}|\left\|T^{n+1}x_n\right\| \\ &+|\beta_n|\,\|T^nx_n\| \\ &\leq \|x_{n+1}-x_n\|+|\beta_{n+1}-\beta_n|\,N_2+|\beta_{n+1}|\,N_3+|\beta_n|\,N_4, \end{aligned}\tag{2.67}$$

where $N_{2,3,4}$ satisfy the following relations:

$$N_2 \geq \sup_{n\geq 1}\|x_n\|\,,\ N_3 \geq \sup_{n\geq 1}\left\|T^{n+1}x_n\right\| \text{ and } N_4 \geq \sup_{n\geq 1}\|T^nx_n\|$$

respectively.
Now substituting (2.67) into (2.66), yields

$$\begin{aligned}\|x_{n+2}-x_{n+1}\| &\leq \alpha_{n+1}\gamma\beta\,\|x_{n+1}-x_n\|+(1-\alpha_{n+1}\tau)\Big(\,\|x_{n+1}-x_n\| \\ &+|\beta_{n+1}-\beta_n|\,N_2+|\beta_{n+1}|\,N_3+|\beta_n|\,N_4\Big) \\ &+|\alpha_{n+1}-\alpha_n|\,N_1 \\ &=(1+\alpha_{n+1}(\gamma\beta-\tau))\,\|x_{n+1}-x_n\|+|\alpha_{n+1}-\alpha_n|\,N_1 \\ &+(1-\alpha_{n+1}\tau)\Big(\,|\beta_{n+1}-\beta_n|\,N_2+|\beta_{n+1}|\,N_3+|\beta_n|\,N_4\Big) \\ &\leq (1-\alpha_{n+1}(\tau-\gamma\beta)\,\|x_{n+1}-x_n\|+ \\ +(1-\alpha_{n+1}\tau)\Big(\,|\beta_{n+1}-\beta_n|&+|\beta_{n+1}|+|\beta_n|+|\alpha_{n+1}-\alpha_n|\,\Big)N_5,\end{aligned}$$

where N_5 choosing appropriately such that $N_5 \geq \max\{N_1, N_2, N_3, N_4\}$.
By lemma (2.3) and (ii), it follows that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0.$$

From equation (2.58), we have,

$$\begin{aligned}\left\|x_{n+1}-T^{\beta_n}x_n\right\| &= \left\|\alpha_n\gamma f(x_n)+(I-\alpha_n\mu G)T^{\beta_n}x_n-T^{\beta_n}x_n\right\| \\ &\leq \alpha_n\left\|\gamma f(x_n)-\mu GT^{\beta_n}x_n\right\| \to 0.\end{aligned}$$

On the other hand,

$$\begin{aligned}\left\|x_{n+1}-T^{\beta_n}x_n\right\| &= \|x_{n+1}-(\beta_n+(1-\beta_n)T^n)x_n\| \\ &= \|(x_{n+1}-x_n)+(1-\beta_n)(x_n-T^nx_n)\| \\ &\geq (1-\beta_n)\,\|x_n-T^nx_n\|-\|x_{n+1}-x_n\|\,,\end{aligned}$$

this implies that

$$\begin{aligned}\|x_n-T^nx_n\| &\leq \frac{\left\|x_{n+1}-T^{\beta_n}x_n\right\|+\|x_{n+1}-x_n\|}{(1-\beta_n)} \\ &\leq \frac{\left\|x_{n+1}-T^{\beta_n}x_n\right\|+\|x_{n+1}-x_n\|}{(1-a)} \to 0.\end{aligned}$$

From the boundedness of $\{x_n\}$, we deduce that $\{x_n\}$ converges weakly. Now assume that $x_n \rightharpoonup p$, by lemma (2.2) and the fact that $\|x_n-T^nx_n\| \to 0$, we obtain $p \in Fix(T^n)$. So, we have

$$\omega_\omega(x_n) \subset Fix(T^n). \tag{2.68}$$

By lemma (2.6) it follows that $(\gamma f - \mu G)$ is strongly monotone, so the variational inequality (2.59) has a unique solution $x^* \in Fix(T^n)$.

Step 4. In this step, we show that

$$\limsup_{n\to\infty} \langle (\gamma f - \mu G)x^*, x_n - x^* \rangle \leq 0. \tag{2.69}$$

The fact that $\{x_n\}$ is bounded, we have $\{x_{n_i}\} \subset \{x_n\}$ such that

$$\limsup_{n\to\infty} \langle (\gamma f - \mu G)x^*, x_n - x^* \rangle = \limsup_{i\to\infty} \langle (\gamma f - \mu G)x^*, x_{n_i} - x^* \rangle \leq 0.$$

Suppose without loss of generality that $x_{n_i} \rightharpoonup x$, from (2.68), it follows that $x \in Fix(T^n)$. Since x^* is the unique solution of (2.58), implies that

$$\begin{aligned}\limsup_{n\to\infty} \langle (\gamma f - \mu G)x^*, x_n - x^* \rangle &= \limsup_{i\to\infty} \langle (\gamma f - \mu G)x^*, x_{n_i} - x^* \rangle . \\ &= \langle (\gamma f - \mu G)x^*, x - x^* \rangle \leq 0.\end{aligned}$$

Step 5. In this step, we show that

$$\lim_{n\to\infty} \|x_n - x^*\| = 0. \tag{2.70}$$

By lemma (2.4) and the fact that f is a contraction, we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \left\|\alpha_n(\gamma f(x_n) - \mu G x^*) + (I - \alpha_n \mu G)T^{\beta_n} x_n - (I - \alpha_n \mu G)x^*\right\|^2 \\ &\leq \left\|(I - \alpha_n \mu G)T^{\beta_n} x_n - (I - \alpha_n \mu G)x^*\right\|^2 \\ &+ 2\alpha_n \langle \gamma f(x_n) - \mu G x^*, x_{n+1} - x^* \rangle \\ &\leq (1 - \alpha_n \tau)^2 \|x_n - x^*\|^2 + 2\alpha_n \gamma \langle f(x_n) - f(x^*), x_{n+1} - x^* \rangle \\ &+ 2\alpha_n \langle \gamma f(x^*) - \mu G x^*, x_{n+1} - x^* \rangle \\ &\leq (1 - \alpha_n \tau)^2 \|x_n - x^*\|^2 + 2\alpha_n \beta \gamma \|x_n - x^*\| \|x_{n+1} - x^*\| \\ &+ 2\alpha_n \langle \gamma f(x^*) - \mu G x^*, x_{n+1} - x^* \rangle \\ &\leq (1 - \alpha_n \tau)^2 \|x_n - x^*\|^2 + \alpha_n \beta \gamma \Big(\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2 \Big) \\ &+ 2\alpha_n \langle \gamma f(x^*) - \mu G x^*, x_{n+1} - x^* \rangle ,\end{aligned}$$

this implies that

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\leq \frac{\Big((1 - \alpha_n \tau)^2 + \alpha_n \beta \gamma\Big) \|x_n - x^*\|^2}{(1 - \alpha_n \gamma \beta)} \\ &+ \frac{2\alpha_n \langle \gamma f(x^*) - \mu G x^*, x_{n+1} - x^* \rangle}{(1 - \alpha_n \gamma \beta)} \\ &\leq \Big(1 - (2\tau - \gamma \beta)\alpha_n\Big) \|x_n - x^*\|^2 + \frac{(\alpha_n \tau)^2}{(1 - \alpha_n \gamma \beta)} \|x_n - x^*\|^2 \\ &+ \frac{2\alpha_n \langle \gamma f(x^*) - \mu G x^*, x_{n+1} - x^* \rangle}{(1 - \alpha_n \gamma \beta)},\end{aligned}$$

this implies that

$$\|x_{n+1} - x^*\|^2 \leq (1 - \gamma_n) \|x_n - x^*\|^2 + \sigma_n,$$

where

$$\begin{aligned}\gamma_n &:= (2\tau-\gamma\beta)\alpha_n \text{ and}\\ \sigma_n &:= \frac{\alpha_n}{(1-\alpha_n\gamma\beta)}\Big(\alpha_n\tau^2\|x_n-x^*\|^2+2\langle\gamma f(x^*)-\mu Gx^*, x_{n+1}-x^*\rangle\Big).\end{aligned}$$

From (3.1 (i)), it follows that

$$\begin{aligned}\lim_{n\to\infty}\gamma_n &= 0,\\ \sum\gamma_n &= \infty,\end{aligned}$$

$$\frac{\sigma_n}{\gamma_n}=\frac{1}{(2\tau-\gamma\beta)(1-\alpha_n\gamma\beta)}\Big(\alpha_n\tau^2\|x_n-x^*\|^2+2\langle\gamma f(x^*)-\mu Gx^*, x_{n+1}-x^*\rangle\Big).$$

Thus $\lim_{n\to\infty}\frac{\sigma_n}{\gamma_n}\leq 0$.

Hence by Lemma (2.3), it follows that $x_n\to x^*$ as $n\to\infty$. □

Corollary 2.65. Let B be a unit ball is a real Hilbert space l_2, and let the mapping $T: B\to B$ be defined by

$$T:(x_1,x_2,x_3,\dots)\to(0,x_1^2,a_2x_2,a_3x_3,\dots),(x_1,x_2,x_3,\dots)\in B,$$

where $\{a_i\}$ is a sequence in $(0,1)$ such that $\prod_{i=2}^{\infty}(a_i)=\frac{1}{2}$. Let, $f,G,\gamma,\{\alpha_n\},\{\beta_n\}$ be as in theorem (3.1). Then the sequence $\{x_n\}$ define by algorithm (2.58), converges strongly to a common fixed point of T^n which solve the variational inequality problem (3.3).

Proof. By example (1.1), it follows that T is $(k,\{\mu\},\{\xi_n\},\phi)$- total asymptotically strict pseudocontraction mapping and uniformly M-Lipschitzian with $M=2\prod_{i=2}^{n}(a_i)$. Hence, the conclusion of this corollary, follows directly from theorem (3.1). □

Corollary 2.66. Let H be a real Hilbert space and $T: H\to H$ be a $(k,\{k_n\})$-asymptotically strict pseudocontraction mapping and uniformly M-Lipschitzian with $M\in(0,1]$. Assume that $Fix(T^n)\neq\emptyset$, and Let f,G,γ $\{\alpha_n\}$ and $\{\beta_n\}$ be as in theorem (3.1). Then, the sequence $\{x_n\}$ generated by algorithm (2.58), converges strongly to a common fixed point of T^n which solve the variational inequality problem (2.59).

Corollary 2.67. [88]Let the sequence $\{x_n\}$ be generated by the mapping

$$x_{n+1}=\alpha_n\gamma f(x_n)+(I-\mu\alpha_n F)Tx_n,$$

where T is nonexpansive, α_n is a sequence in (0,1) satisfying the following conditions:

$$\begin{cases}\text{(i) } \lim_{n\to\infty}\alpha_n=0,\ \sum\alpha_n=\infty;\\ \text{(ii) } \sum|\alpha_{n+1}-\alpha_n|<\infty,\ \sum|\beta_{n+1}-\beta_n|<\infty;\\ \text{(iii) } 0\leq\max_i k_i\leq\beta_n<a<1,\forall n\geq 0.\end{cases}\tag{2.71}$$

It was proved in [88] that $\{x_n\}$ converged strongly to the common fixed point x^* of T, which is the solution of variational inequality problem

$$\langle (\gamma f - \mu F)x^*, x - x^* \rangle \leq 0, \forall x \in Fix(T). \tag{2.72}$$

Proof. Take n=1, $k = \mu_n = \xi_n = 0$ and $F = G$ in theorem (3.1). Therefore all the conditions in theorem (3.1) are satisfied. Hence the conclusion of this corollary follows directly from theorem (3.1). □

Corollary 2.68. [87]Let the sequence $\{x_n\}$ be generated by

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Tx_n,$$

where T is nonexpansive and the sequence $\alpha_n \subset (0, 1)$ satisfy the conditions in equation (2.57). Then it was proved in [87] that $\{x_n\}$ converged strongly to x^* which solve the variational inequality

$$\langle (\gamma f - A)x^*, x - x^* \rangle \leq 0, \forall x \in Fix(T). \tag{2.73}$$

Proof. Take n=1, $\mu_n = \xi_n = 0$ and $\mu = 1$ and $G = A$ in theorem (3.1). Therefore all the conditions in theorem (3.1) are satisfied. Hence the conclusion of this corollary follows directly from theorem (3.1). □

Corollary 2.69. [85] Let the sequence $\{x_n\}$ be generated by

$$x_{n+1} = Tx_n - \mu \lambda_n F(Tx_n),$$

where T is nonexpansive mapping on H, F is L-Lipschitzian and η-strongly monotone with $L > 0, \eta > 0$ and $0 < \mu < \frac{2\eta}{L^2}$, if the sequence $\lambda_n \subset (0, 1)$ satisfies the following conditions:

$$\begin{cases} \text{(i) } \lim\limits_{n\to\infty} \lambda_n = 0, \sum \lambda_n = \infty; \\ \text{(ii) either} \sum |\lambda_{n+1} - \lambda_n| = 0 \text{ or } \lim\limits_{n\to\infty} \dfrac{\lambda_{n+1}}{\lambda_n} = 1. \end{cases} \tag{2.74}$$

Then, it was proved by Yamada in [85] that $\{x_n\}$ converged strongly to the unique solution of the variational inequality

$$\langle Fx^*, x - x^* \rangle \geq 0, \forall x \in Fix(T). \tag{2.75}$$

Proof. Take $n = 1$, $k = \mu_n = \xi_n = 0$ and also take $\gamma = 0$, $\beta_n = 0$ and $G = F$. Therefore all the conditions in theorem (3.1) are satisfied. Hence the result follows directly from theorem (3.1). □

3. A note on the Split Equality Fixed Point Problems in Hilbert Spaces

In this section, we propose the split feasibility and fixed point equality problems (SFFPEP) and split common fixed point equality problems (SCFPEP). Furthermore, we formulate and analyse the algorithms for solving these problems for the class of quasi-nonexpansive mappings in Hilbert spaces. In the end, we study the convergence results of the proposed algorithms.

3.1. **Problem Formulation.** The split feasibility and fixed point equality problems (in short, SFFPEP) formulated as follows:

$$\text{Find } x^* \in C \cap Fix(U) \text{ and } y^* \in Q \cap Fix(T) \text{ such that } Ax^* = By^*. \tag{3.1}$$

While the split common fixed point equality problems (in short, SCFPEP) obtained as follows:

$$\text{Find } x^* \in \bigcap_{i=1}^{N} Fix(U_i) \text{ and } y^* \in \bigcap_{j=1}^{M} Fix(T_j) \text{ such that } Ax^* = By^*, \tag{3.2}$$

where $U_{i=1} : H_1 \to H_1, i = 1, 2, 3, ..., N$, and $T_{j=1} : H_2 \to H_2, j = 1, 2, 3, ..., M$, are quasi-nonexpansive mappings with $Fix(U_i) \neq \emptyset$ and $Fix(T_j) \neq \emptyset$, respectively, $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are bounded linear operators.

Note that if $C := Fix(U)$, $Q := Fix(T)$, $H_2 = H_3$ and $B = I$. Then Problem (3.1) reduces to the following problems:

$$\text{Find } x^* \in C \text{ and } y^* \in Q \text{ such that } Ax^* = By^*, \tag{3.3}$$

and

$$x^* \in C \cap Fix(U) \text{ such that } Ax^* \in Q \cap Fix(T). \tag{3.4}$$

Equation (3.3) and (3.4) are called the split equality fixed point problems (SEFPP) and split feasibility and fixed point problems (SFFPP), respectively. In the light of this, it is worth to mention here that the SFFPEP generalizes the SFP, SFFPP, and SEFPP. Therefore, the results and conclusions that are true for the SFFPEP continue to hold for these problems (SFP, SFFPP, and SEFPP), and it shows the significance and the range of applicability of the SFFPEP.

Furthermore, Problem (3.2) reduces to Problem (2.8) as $H_2 = H_3$ and $B = I$. This shows that the SCFPEP generalizes the SCFPP. Therefore, the results and conclusions that are true for the SCFPEP continue to hold for the SCFPP.

We denote the solution of sets SFFPEP (3.1) and SCFPEP (3.2) by

$$\Phi = \Big\{ x^* \in C \cap Fix(U) \text{ and } y^* \in Q \cap Fix(T) \text{ such that } Ax^* = By^* \Big\}, \tag{3.5}$$

and

$$\Psi = \Big\{ x^* \in \bigcap_{i=1}^{N} Fix(U_i) \text{ and } y^* \in \bigcap_{j=1}^{M} Fix(T_j) \text{ such that } Ax^* = By^* \Big\}, \tag{3.6}$$

respectively. In sequel, we assume that Φ and Ψ are nonempty.

3.2. **Preliminaries.** In this section, we present some lemmas used in proving our main result.

Lemma 3.1. Let $C \subset H$ and $\{x_n\}$ be a sequence in H such that the following conditions are satisfied:

(i) For each $x \in C$, $\lim_{n\to\infty} \|x_n - x\|$ exists,

(ii) Any weak-cluster point of the sequence $\{x_n\}$ belongs to C.

Then, there exists $y \in C$ such that $\{x_n\}$ converges weakly to y.

For the proof, see [83] and references therein.

Lemma 3.2. Let $T_i : H \to H$, for i=1,2,3,...,N be N-quasi-nonexpansive mappings. Defined $U = \sum_{i=1}^{N} \delta_i U_{\beta_i}$, where $U_{\beta_i} = (1-\beta_i)I + \beta_i T_i$, and $\delta_i \in (0,1)$ such that $\sum_{i=1}^{N} \delta_i = 1$. Then

(i) U is a quasi-nonexpansive mapping,
(ii) $Fix(U) = \bigcap_{i=1}^{N} Fix(U_{\beta_i}) = \bigcap_{i=1}^{N} Fix(T_i)$,
(iii) in addition, if $(T_i - I)$ for i=1,2,3,...,N is demiclosed at zero, then $(U - I)$ is also demiclosed at zero.

For the proof, see Li and He [30] and the references therein.

3.3. **The Split Feasibility and Fixed Point Equality Problems for Quasi-Nonexpansive Mappings in Hilbert Spaces.** To approximate the solution of the split feasibility and fixed point equality problems (3.5), we make the following assumptions:

(B_1) $U : H_1 \to H_1$ and $T : H_2 \to H_2$ are quasi-nonexpansive mappings with $Fix(U) \neq \emptyset$ and $Fix(T) \neq \emptyset$, respectively.
(B_2) $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are bounded linear operators with their adjoints A^* and B^*, respectively.
(B_3) $(U - I)$ and $(T - I)$ are demiclosed at zero.
(B_4) P_C and P_Q are metric projection of H_1 and H_2 onto C and Q, respectively.
(B_5) For arbitrary $x_1 \in H_1$ and $y_1 \in H_2$, define a sequence $\{(x_n, y_n)\}$ by:

$$\begin{cases} z_n = P_C(x_n - \lambda_n A^*(Ax_n - By_n)), \\ w_n = (1-\beta_n)z_n + \beta_n U(z_n), \\ x_{n+1} = (1-\alpha_n)z_n + \alpha_n U(w_n), \\ \\ u_n = P_Q(y_n + \lambda_n B^*(Ax_n - By_n)), \\ r_n = (1-\beta_n)u_n + \beta_n T(u_n), \\ y_{n+1} = (1-\alpha_n)u_n + \alpha_n T(r_n), \forall n \geq 1, \end{cases} \tag{3.7}$$

where $0 < a < \beta_n < 1$, $0 < b < \alpha_n < 1$, and $\lambda_n \in \left(0, \frac{2}{L_1 + L_2}\right)$, where $L_1 = A^*A$ and $L_2 = B^*B$, respectively.

We are now in the position to state and prove the main result of this chapter.

Theorem 3.3. Suppose that assumption $(B_1) - (B_5)$ are satisfied, also assume that the solution set $\Phi \neq \emptyset$. Then $(x_n, y_n) \rightharpoonup (x^*, y^*) \in \Phi$.

Proof. Let $(x^*, y^*) \in \Phi$. By (3.7), we have

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|(1-\alpha_n)(z_n - x^*) + \alpha_n(Uw_n - x^*)\|^2 \\ &= (1-\alpha_n)\|z_n - x^*\|^2 + \alpha_n \|Uw_n - x^*\|^2 - \alpha_n(1-\alpha_n)\|Uw_n - z_n\|^2 \\ &\leq (1-\alpha_n)\|z_n - x^*\|^2 + \alpha_n \|w_n - x^*\|^2 \\ &- \alpha_n(1-\alpha_n)\|Uw_n - z_n\|^2 . \end{aligned} \tag{3.8}$$

On the other hand,

$$\begin{aligned}\|w_n - x^*\|^2 &= \|(1-\beta_n)(z_n - x^*) + \beta_n(Uz_n - x^*)\|^2 \\ &= (1-\beta_n)\|z_n - x^*\|^2 + \beta_n\|Uz_n - x^*\|^2 - \beta_n(1-\beta_n)\|Uz_n - z_n\|^2 \\ &\leq \|z_n - x^*\|^2 - \beta_n(1-\beta_n)\|Uz_n - z_n\|^2 . \end{aligned} \tag{3.9}$$

Substituting (3.9) into (3.8), we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 \leq{}& (1-\alpha_n)\|z_n - x^*\|^2 + \alpha_n\|z_n - x^*\|^2 \\ & - \alpha_n\beta_n(1-\beta_n)\|Uz_n - z_n\|^2 - \alpha_n(1-\alpha_n)\|Uw_n - z_n\|^2 . \end{aligned} \tag{3.10}$$

On the Other hand,

$$\begin{aligned}\|z_n - x^*\|^2 &= \|P_C(x_n - \lambda_n A^*(Ax_n - By_n)) - P_C(x^*)\|^2 \\ &\leq \|x_n - \lambda_n A^*(Ax_n - By_n) - x^*\|^2 \\ &= \|x_n - x^*\|^2 - 2\lambda_n\langle Ax_n - Ax^*, Ax_n - By_n\rangle \\ &+ \lambda_n^2 L_1\|Ax_n - By_n\|^2 . \end{aligned} \tag{3.11}$$

Substituting (3.11) into (3.10), we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 \leq{}& \|x_n - x^*\|^2 - 2\lambda_n\langle Ax_n - Ax^*, Ax_n - By_n\rangle + \lambda_n^2 L_1\|Ax_n - By_n\|^2 \\ & - \alpha_n\beta_n(1-\beta_n)\|U(z_n) - z_n\|^2 - \alpha_n(1-\alpha_n)\|Uw_n - z_n\|^2 . \end{aligned} \tag{3.12}$$

Similarly, the second equation of Equation (3.7) gives

$$\begin{aligned}\|y_{n+1} - y^*\|^2 \leq{}& \|y_n - y^*\|^2 + 2\lambda_n\langle By_n - By^*, Ax_n - By_n\rangle + \lambda_n^2 L_2\|Ax_n - By_n\|^2 \\ & - \alpha_n\beta_n(1-\beta_n)\|T(u_n) - u_n\|^2 - \alpha_n(1-\alpha_n)\|Tr_n - u_n\|^2 . \end{aligned} \tag{3.13}$$

By (3.12), (3.13) and noticing that $Ax^* = By^*$, we deduce that

$$\begin{aligned}\|x_{n+1} - x^*\|^2 + \|y_{n+1} - y^*\|^2 \leq{}& \|x_n - x^*\|^2 + \|y_n - y^*\|^2 - 2\lambda_n\|Ax_n - By_n\|^2 \\ & + \lambda_n^2(L_1 + L_2)\|Ax_n - By_n\|^2 \\ & - \alpha_n\beta_n(1-\beta_n)\|U(z_n) - z_n\|^2 \\ & - \alpha_n\beta_n(1-\beta_n)\|T(u_n) - u_n\|^2 . \end{aligned} \tag{3.14}$$

Thus, we deduce that

$$\begin{aligned}\Phi_{n+1} \leq{}& \Phi_n - \lambda_n(2 - \lambda_n(L_1 + L_2))\|Ax_n - By_n\|^2 \\ & - \alpha_n\beta_n(1-\beta_n)\|U(z_n) - z_n\|^2 - \alpha_n\beta_n(1-\beta_n)\|T(u_n) - u_n\|^2 , \end{aligned} \tag{3.15}$$

where

$$\Phi_n := \|x_n - x^*\|^2 + \|y_n - y^*\|^2 .$$

Thus, $\{\Phi_n\}$ is a non-increasing sequence and bounded below by 0, therefore, it converges.

From (3.15) and the fact that $\{\Phi_n\}$ converges, we deduce that

$$\lim_{n\to\infty}\|Ax_n - By_n\| = 0, \tag{3.16}$$

$$\lim_{n\to\infty}\|Uz_n - z_n\| = 0 \text{ and } \lim_{n\to\infty}\|Tu_n - u_n\| = 0. \tag{3.17}$$

Furthermore, since $\{\Phi_n\}$ converges, this ensures that $\{x_n\}$ and $\{y_n\}$ also converges. This further implies that $x_n \rightharpoonup x$ and $y_n \rightharpoonup y$ for some $(x,y) \in \Phi$.

Now, $(x,y) \in \Phi$, implies that $x \in C \cap Fix(U)$ and $y \in Q \cap Fix(T)$ such that $Ax = By$. The fact that $x_n \rightharpoonup x$ and $\lim_{n\to\infty} \|Ax_n - By_n\| = 0$ together with

$$z_n = P_C(x_n - \lambda_n A^*(Ax_n - By_n)),$$

we deduce that $z_n \rightharpoonup P_C x$. Since $x \in C$, by projection theorem, we obtain that $P_C x = x$. Hence, $z_n \rightharpoonup x$.

Similarly, The fact that $y_n \rightharpoonup y$ and $\lim_{n\to\infty} \|Ax_n - By_n\| = 0$ together with

$$u_n = P_Q(y_n + \lambda_n B^*(Ax_n - By_n)),$$

we deduce that $u_n \rightharpoonup P_Q y$. Since $y \in Q$, by projection theorem, we obtain that $P_Q y = y$. Hence, $u_n \rightharpoonup y$.

Now, $z_n \rightharpoonup x$, $\lim_{n\to\infty} \|Uz_n - z_n\| = 0$, and together with the demiclosed of $(U - I)$ at zero, we deduce that $Ux = x$, this implies that $x \in Fix(U)$.

On the other hand, $u_n \rightharpoonup y$ and $\lim_{n\to\infty} \|Tu_n - u_n\| = 0$ together with the demiclosed of $(T - I)$ at zero, we deduce that $Ty = y$, this implies that $y \in Fix(T)$.

Since $z_n \rightharpoonup x$, $u_n \rightharpoonup y$ and the fact that A and B are bounded linear operators, we have

$$Az_n \rightharpoonup Ax \text{ and } Bu_n \rightharpoonup By,$$

this implies that

$$Az_n - Bu_n \rightharpoonup Ax - By,$$

which turn to implies that

$$\|Ax - By\| \leq \liminf_{n\to\infty} \|Az_n - Bu_n\| = 0,$$

which further implies that $Ax = By$. Noticing that $x \in C$, $x \in Fix(U)$, $y \in Q$ and $y \in Fix(T)$, we have that $x \in C \cap Fix(U)$ and $y \in Q \cap Fix(T)$. Hence, we conclude that $(x,y) \in \Phi$.

Summing up, we have proved that:

(i) for each $(x^*, x^*) \in \Phi$, the $\lim_{n\to\infty} \left(\|x_n - x^*\|^2 + \|y_n - y^*\|^2 \right)$ exist;
(ii) the weak cluster of the sequence (x_n, y_n) belongs to Φ.

Thus, by Lemma (3.1) we conclude that the sequences (x_n, y_n) converges weakly to $(x^*, x^*) \in \Phi$. This completes the proof. $\square$

Theorem 3.4. Suppose that all the hypothesis of Theorem 3.3 is satisfied. Also, assume that U and T are semi-compacts, then $(x_n, y_n) \to (x^*, y^*) \in \Phi$.

Proof. As in the proof of Theorem 3.3, $\{u_n\}$ and $\{z_n\}$ are bounded, by (3.17) and the fact that U and T are semi-compacts, then there exists sub-sequences $\{u_{n_k}\}$ and $\{z_{n_k}\}$ (suppose without loss of generality) of $\{u_n\}$ and $\{z_n\}$ such that $u_{n_k} \to x$

and $z_{n_k} \to y$. Since, $u_n \rightharpoonup x^*$ and $z_n \rightharpoonup y^*$, we have $x = x^*$ and $y = y^*$. By (3.16) and the fact that $u_{n_k} \to x^*$ and $z_{n_k} \to y^*$, we have

$$\lim_{n\to\infty} \|Ax^* - Ay^*\| = \lim_{n\to\infty} \|Au_{n_k} - Bz_{n_k}\| = 0, \tag{3.18}$$

which tends to imply that $Ax^* = Ay^*$. Hence $(x^*, y^*) \in \Phi$. Thus, the iterative algorithm of Theorem 3.3 conveges strongly to the solution of Problem 3.5.

□

3.4. **The Split Common Fixed Point Equality Problems for Quasi - Nonexpansive Mappings in Hilbert Spaces.** To approximate the solution of split common fixed point equality problems, we make the following assumptions:

- (A_1) $T_1, T_2, T_3, ..., T_N : H_1 \to H_1$ and $U_1, U_2, U_3, ..., U_M : H_2 \to H_2$ are quasi-nonexpansive mappings with $\bigcap_{i=1}^N Fix(T_i) \neq \emptyset$ and $\bigcap_{i=1}^N Fix(U_j) \neq \emptyset$, respectively.
- (A_2) $(T_i - I)$, for i=1,2,3,...,N and $(U_j - I)$, for i=1,2,3,...,M are demiclosed at zero.
- (A_3) $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are bounded linear operators with their adjoints A^* and B^*, respectively.
- (A_4) For arbitrary $x_1 \in H_1$ and $y_1 \in H_2$, define $\{(x_n, y_n)\}$ by:

$$\begin{cases} z_n = x_n - \lambda_n A^*(Ax_n - By_n), \\ w_n = (1-\beta_n)z_n + \beta_n \sum_{j=1}^M \delta_j U_{\gamma_j}(z_n), \\ x_{n+1} = (1-\alpha_n)z_n + \alpha_n \sum_{j=1}^M \delta_j U_{\gamma_j}(w_n), \\ \\ u_n = y_n + \lambda_n B^*(Ax_n - By_n), \\ r_n = (1-\beta_n)u_n + \beta_n \sum_{i=1}^N \lambda_i T_{\tau_i}(u_n), \\ y_{n+1} = (1-\alpha_n)u_n + \alpha_n \sum_{i=1}^N \lambda_i T_{\tau_i}(r_n), \forall n \geq 1, \end{cases} \tag{3.19}$$

where $U_{\gamma_j} = (1-\gamma_j)I + \gamma_j U_j$ and $\gamma_j \in (0, 1)$, for j=1,2,3,...,M, $T_{\tau_i} = (1-\tau_i)I + \tau_i T_i$, and $\tau_i \in (0, 1)$, for i=1,2,3,...,N, $\sum_{j=1}^M \delta_j = 1$ and $\sum_{i=1}^N \lambda_i = 1$, $0 < a < \beta_n < 1$, $0 < b < \alpha_n < 1$ and $\lambda_n \in \left(0, \frac{2}{L_1+L_2}\right)$ where $L_1 = A^*A$ and $L_2 = B^*B$, respectively.

Theorem 3.5. Suppose that conditions $(A_1) - (A_4)$ above are satisfied, also, assume that the solution set $\Psi \neq \emptyset$. Then $(x_n, y_n) \rightharpoonup (x^*, y^*) \in \Psi$.

Proof. Let $(x^*, y^*) \in \Psi$ and $U = \sum_{j=1}^M \delta_j U_{\gamma_j}$ and $T = \sum_{i=1}^N \lambda_i T_{\tau_i}$. By Lemma 3.2, we deduce that U and T are quasi nonexpansive mappings, $Fix(U) = \bigcap_{j=1}^M Fix(U_{\delta_j}) = \bigcap_{j=1}^M Fix(U_j)$ and $Fix(T) = \bigcap_{i=1}^N Fix(T_{\tau_i}) = \bigcap_{i=1}^N Fix(T_i)$, respectively. By Algorithm (3.19), we deduce the following algorithm.

$$\begin{cases} z_n = x_n - \lambda_n A^*(Ax_n - By_n), \\ w_n = (1-\beta_n)z_n + \beta_n U(z_n), \\ x_{n+1} = (1-\alpha_n)z_n + \alpha_n U(w_n), \\ \\ u_n = y_n + \lambda_n B^*(Ax_n - By_n), \\ r_n = (1-\beta_n)u_n + \beta_n T(u_n), \\ y_{n+1} = (1-\alpha_n)u_n + \alpha_n T(r_n), \forall n \geq 1. \end{cases} \tag{3.20}$$

Thus, all the hypothesis of Theorem 3.3 is satisfied. Hence the proof of this theorem follows directly from Theorem 3.3.

□

Corollary 3.6. Suppose that conditions $(B_1)-(B_5)$ are satisfied and let $\{(x_n, y_n)\}$ be the sequence generated by Algorithm (3.7). Assume that $\Phi \neq \emptyset$, and let U and T be the firmly of quasi-nonexpansive mappings. Then the sequence $\{(x_n, y_n)\}$ generated by Algorithm (3.7) converges weakly to the solution of Problem (3.5).

Corollary 3.7. Suppose that conditions $(B_1)-(B_4)$ are satisfied and let the sequence $\{(x_n, y_n)\}$ be generated by

$$\begin{cases} z_n = x_n - \lambda_n A^*(Ax_n - By_n), \\ x_{n+1} = (1-\alpha_n)z_n + \alpha_n U(z_n), \\ \\ u_n = y_n + \lambda_n B^*(Ax_n - By_n), \\ y_{n+1} = (1-\alpha_n)u_n + \alpha_n T(y_n), \forall n \geq 0, \end{cases} \tag{3.21}$$

where $0 < a < \beta_n < 1$, and $\lambda_n \in \left(0, \frac{2}{L_1+L_2}\right)$, where $L_1 = A^*A$ and $L_2 = B^*B$. Assume that $\Phi \neq \emptyset$. Then the sequence $\{(x_n, y_n)\}$ generated by Algorithm (3.21) converges weakly to the solution of SEFPP (3.3).

Proof. Trivially, Algorithm (3.7) reduces to Algorithm (3.21) as $\beta = 0$, $P_C = P_Q = I$ and SFFPEP (3.4) reduces to SEFPP (3.3) as $C := Fix(U)$ and $Q := Fix(T)$. Therefore, all the hypothesis of Theorem 3.3 is satisfied. Hence, the proof of this corollary follows directly from Theorem 3.3. □

Corollary 3.8. Suppose that conditions $(A_1)-(A_4)$ are satisfied, and let the sequence $\{(x_n, y_n)\}$ be defined by Algorithm (3.19). Assume that $\Psi \neq \emptyset$ and let U and T be firmly quasi-nonexpansive mappings, where $U = \sum_{j=1}^{M} \delta_j U_{\gamma_j}$ and $T = \sum_{i=1}^{N} \lambda_i T_{\tau_i}$. Then $(x_n, y_n) \rightharpoonup (x^*, x^*) \in \Psi$.

4. Numerical Example

In this section, we give the numerical examples that illustrates our theoretical results.

Example 4.1. Let $H_1 = \Re$ with the inner product defined by $\langle x, y\rangle = xy$ for all $x, y \in \Re$ and $\|.\|$ stands for the corresponding norm. Let $C := [0, \infty)$ and $Q := [0, \infty)$. Defined $T : C \to \Re$ and $S : Q \to \Re$ by $Tx = \frac{x^2+5}{1+x}$, $\forall x \in C$ and $Sx = \frac{x+5}{5}$, $\forall x \in Q$. Then T and S are quasi nonexpansive mappings.

Proof. □

Trivially, $Fix(T) = 5$ and $Fix(S) = \frac{5}{4}$.
Now,

$$\begin{aligned} |Tx - 5| &= \left| \frac{x^2+5}{1+x} - 5 \right| \\ &= \frac{x}{1+x} |x-5| \\ &\leq |x-5|. \end{aligned}$$

On the other hand,

$$\begin{aligned}\left|Sx-\frac{5}{4}\right| &= \left|\frac{x+5}{5}-\frac{5}{4}\right| \\ &= \frac{1}{5}\left|x-\frac{5}{4}\right| \\ &\leq \left|x-\frac{5}{4}\right|.\end{aligned}$$

Thus, T and S are quasi-nonexpansive mappings.

Example 4.2. Let $H_1 = \Re$, $H_2 = \Re$, $C := [0,\infty)$, and $Q := [0,\infty)$ be subset of H_1 and H_2, respectively. Defined $T : C \to C$ by $Tx = \frac{x+2}{3}$ $\forall x \in C$, and $U : Q \to Q$ by

$$Ux = \begin{cases} \frac{2x}{x+1}, \forall x \in (1,+\infty) \\ 0,\ \forall x \in [0,1]. \end{cases} \tag{4.1}$$

Then, U and T are quasi nonexpansive mappings.

Proof. Trivially, $Fix(T) = 1$ and $Fix(U) = 1$.
Now,

$$\begin{aligned}|Tx-1| &= \left|\frac{x+2}{3}-1\right| \\ &\leq |x-1|.\end{aligned}$$

And also,

$$\begin{aligned}|Ux-1| &= \frac{1}{1+x}|x-1| \\ &\leq |x-1|.\end{aligned}$$

Thus, U and T are quasi nonexpansive mappings. □

The following example is a particular case of Theorem 3.3

Example 4.3. Let $H_1 = \Re$ with the inner product defined by $\langle x, y\rangle = xy$ for all $x, y \in \Re$ and $\|.\|$ stands for the corresponding norm. Let $C := [0,\infty)$ and $Q := [0,\infty)$. Defined $U : C \to \Re$ and $T : Q \to \Re$ by $Ux = \frac{x^2+5}{1+x}$, $\forall x \in C$ and $Tx = \frac{x+5}{5}$, $\forall x \in Q$. And also let $P_C = P_Q = I$, $Ax = x$, $By = 4y$, $\lambda_n = 1$, $\alpha_n = \frac{1}{5}$, $\beta_n = \frac{1}{8}$ and $\{(x_n, y_n)\}$ be the sequence generated by Algorithm (3.7). That is

$$\begin{cases} x_0 \in C \text{ and } y_0 \in Q, \\ z_n = P_C(x_n - A^*(x_n - 4y_n)), \\ w_n = (1-\frac{1}{8})z_n + \frac{1}{8}U(z_n), \\ x_{n+1} = (1-\frac{1}{5})z_n + \frac{1}{5}U(w_n), \\ \\ u_n = P_Q(y_n + B^*(x_n - 4y_n)), \\ r_n = (1-\frac{1}{8})u_n + \frac{1}{8}T(u_n), \\ y_{n+1} = (1-\frac{1}{5})u_n + \frac{1}{5}T(r_n), \forall n \geq 0. \end{cases} \tag{4.2}$$

Then (x_n, y_n) converges to $(5, 5/4) \in \Psi$.

Proof. By Example 4.1 U and T are quasi-nonexpansive mappings. Clearly, A and B are bounded linear operator on $\Re$ with $A = A^* = 1$ and $B = B^* = 4$, respectively. Furthermore, it is easy to see that $Fix(U) = 5$ and $Fix(T) = \frac{5}{4}$. Hence,

$$\Psi = \Big\{5 \in C \cap Fix(U) \text{ and } 5/4 \in Q \cap Fix(T) \text{ such that } A(5) = B(5/4)\Big\}.$$

Simplifying Algorithm (4.2), we obtain the following algorithm.

$$\begin{cases} x_0 \in C \text{ and } y_0 \in Q, \\ z_n = x_n, \\ w_n = \frac{7}{8}z_n + \frac{1}{8}(\frac{z_n^2+5}{z_n+1}), \\ x_{n+1} = \frac{4}{5}z_n + \frac{1}{5}(\frac{w_n^2+5}{w_n+1}), \\ \\ u_n = y_n, \\ r_n = \frac{7}{8}u_n + \frac{1}{8}(\frac{u_n+5}{5}), \\ y_{n+1} = \frac{4}{5}u_n + \frac{1}{5}(\frac{r_n+5}{5}), \forall n \geq 0. \end{cases} \tag{4.3}$$

□

We used Maple and obtained the numerical values of Algorithm 4.3 in the tables below.

TABLE 1. Shows the numerical values of Example 4.3 Algorithm (4.3), starting with the initial values $x_0 = 10.$ and $y_0 = 15$

n	x_n	y_n
0	10.00000000	15.00000000
1	9.898293685	12.74500000
2	9.797736851	10.85982000
3	9.698337655	9.283809520
.	.	.
.	.	.
.	.	.
248	5.001051418	1.250000002
249	5.001012726	1.250000002
250	5.000975458	1.250000002

FIGURE 1. Shows the convergence of Example 4.3 Algorithm (4.3), starting with the initial value $x_0 = 10$ and $y_0 = 15$.

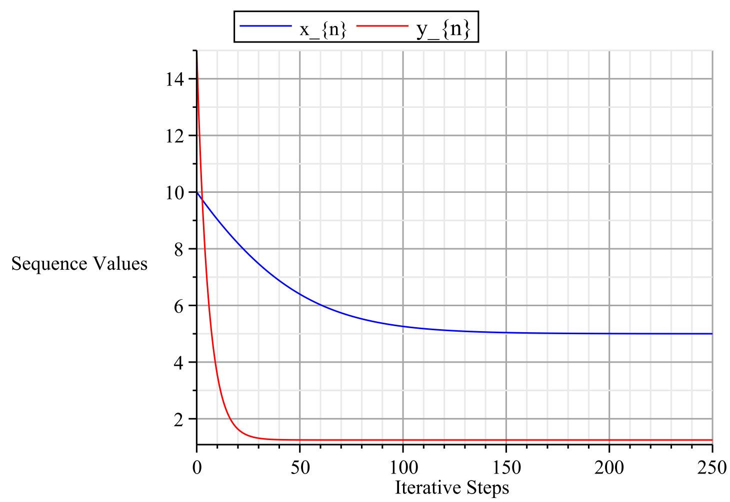
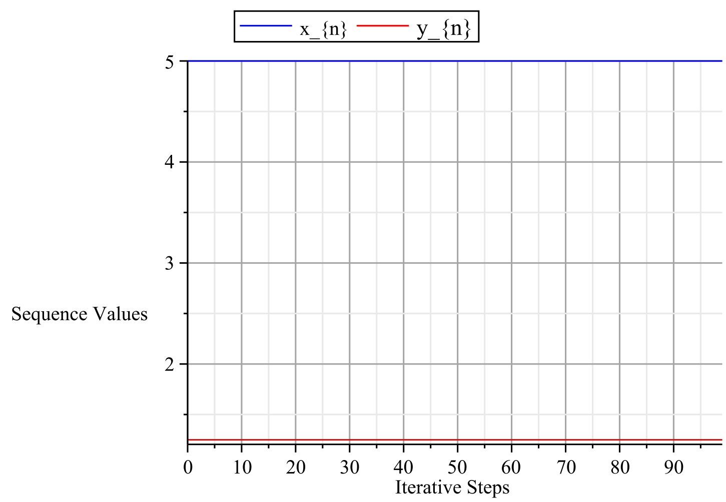


TABLE 2. Shows the numerical values of Example 4.3 Algorithm (4.3), starting with the initial values $x_0 = 5$ and $y_0 = 1.25$.

n	x_n	y_n
0	5.000000000	1.250000000
1	5.000000000	1.250000000
2	5.000000000	1.250000000
.	.	.
.	.	.
.	.	.
98	5.000000000	1.250000000
99	5.000000000	1.250000000
100	5.000000000	1.250000000

FIGURE 2. Shows the convergence of Example 4.3 Algorithm (4.3), starting with the initial value $x_0 = 5$ and $y_0 = 1.25$



The following example is a particular case of Theorem 3.5

Example 4.4. Let $H_1 = \Re$ and $H_2 = \Re$, $C := [0, \infty)$ and $Q := [0, \infty)$ be subset of H_1 and H_2, respectively. Define $T : C \to C$ by $Tx = \frac{x+2}{3}$ $\forall x \in C$, and $U : Q \to Q$ by

$$Ux = \begin{cases} \frac{2x}{x+1}, \forall x \in (1, +\infty) \\ 0, \ \forall x \in [0, 1]. \end{cases} \tag{4.4}$$

Let also $\lambda_n = 1$, $Ax = x$, $By = y$, $\gamma_j = \frac{1}{3}$, $\tau_i = \frac{1}{5}$, $\alpha_n = \frac{1}{7}$ and $\beta_n = \frac{1}{9}$. The sequence $\{(x_n, y_n)\}$ defined by Algorithm 3.19 can be written as follows:

$$\begin{cases} z_n = x_n - A^*(Ax_n - By_n), \\ w_n = \frac{8}{9}z_n + \frac{1}{9}\left(\frac{2z_n}{3} + \frac{2z_n}{3(z_n+1)}\right), \\ x_{n+1} = \frac{6}{7}z_n + \frac{1}{7}\left(\frac{2w_n}{3} + \frac{2w_n}{3(w_n+1)}\right), \\ \\ u_n = y_n + B^*(Ax_n - By_n), \\ r_n = \frac{8}{9}u_n + \frac{1}{9}\left(\frac{4u_n}{5} + \frac{u_n+2}{15}\right), \\ y_{n+1} = \frac{6}{7}u_n + \frac{1}{7}\left(\frac{4r_n}{5} + \frac{r_n+2}{15}\right), \forall n \geq 1. \end{cases} \tag{4.5}$$

Then (x_n, y_n) converges to $(1, 1) \in \Psi$.

Proof. By Example 4.2, U and T are quasi nonexpansive mappings with $Fix(U) = 1$ and $Fix(T) = 1$, respectively. Clearly, A, B are bounded linear on $\Re$, $A = A^* = 1$ and $B = B^* = 1$. Hence,

$$\Psi = \{1 \in Fix(T) \text{ and } 1 \in Fix(U) \text{ such that } A(1) = B(1)\}.$$

Simplifying Algorithm (4.5), we have

$$\begin{cases} z_n = y_n, \\ w_n = \frac{8}{9}z_n + \frac{1}{9}\left(\frac{2z_n}{3} + \frac{2z_n}{3(z_n+1)}\right), \\ x_{n+1} = \frac{6}{7}z_n + \frac{1}{7}\left(\frac{2w_n}{3} + \frac{2w_n}{3(w_n+1)}\right), \\ \\ u_n = x_n, \\ r_n = \frac{8}{9}u_n + \frac{1}{9}\left(\frac{4u_n}{5} + \frac{u_n+2}{15}\right), \\ y_{n+1} = \frac{6}{7}u_n + \frac{1}{7}\left(\frac{4r_n}{5} + \frac{r_n+2}{15}\right), \forall n \geq 1. \end{cases} \tag{4.6}$$

□

TABLE 3. Shows the numerical values of Example 4.4 Algorithm (4.5), starting with the initial values $x_0 = 5$ and $y_0 = 5$.

n	x_n	y_n
0	5.000000000	5.000000000
1	4.916472663	4.760850019
2	4.834689530	4.537828465
3	4.754614179	4.329771078
.	.	.
.	.	.
.	.	.
148	1.176058095	1.007392532
149	1.172381679	1.007122340

FIGURE 3. Shows the convergence of Example 4.4 Algorithm (4.5), starting with the initial value $x_0 = 5$ and $y_0 = 5$.

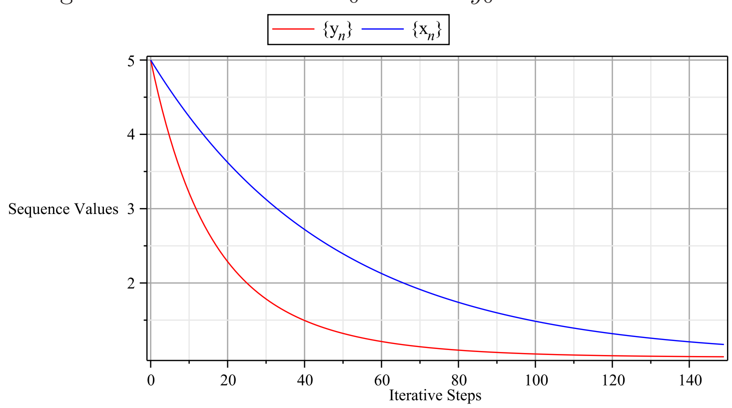
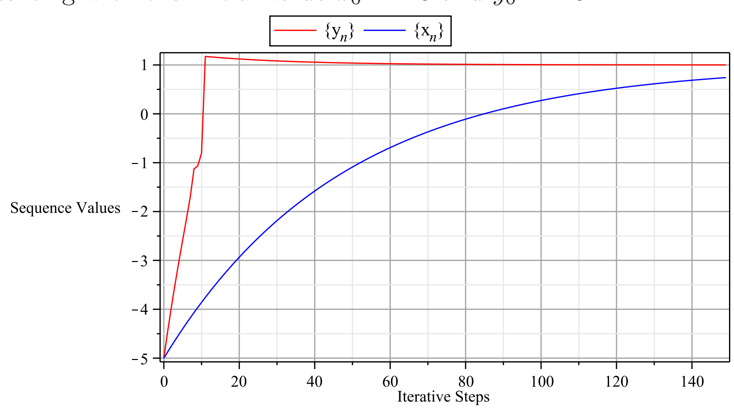


TABLE 4. Shows the numerical values of Example 4.4 Algorithm (4.5), starting with the initial values $x_0 = -5.$ and $y_0 = -5$

n	x_n	y_n
0	-5.000000000	-5.000000000
1	-4.460475401	-4.874708995
2	-3.953349994	-4.752034296
.	-3.474475616	-4.631921270
.	.	.
.	.	.
.	.	.
148	1.001346412	0 .7359128532
149	1.001297344	0.7414274772

FIGURE 4. Shows the convergence of Example 4.4 Algorithm (4.5), starting with the initial value $x_0 = -5$ and $y_0 = -5$.



5. The split feasibility and fixed point problems for quasi-nonexpansive mappings in Hilbert spaces

In this section, we propose Ishikawa-type extra-gradient algorithms for solving the split feasibility and fixed point problems. Under some suitable assumptions imposed on some parameters and operators involved, we prove the strong convergence theorems of these algorithms.

5.1. **Problem Formulation.** The split feasibility and fixed point problems (SFFPP) required to find a vector

$$x^* \in C \cap Fix(T_1) \text{ such that } Ax^* \in Q \cap Fix(T_2), \tag{5.1}$$

where $T_1 : H_1 \to H_1$ and $T_2 : H_2 \to H_2$ are quasi-nonexpansive mappings, and $A : H_1 \to H_2$ is a bounded linear operator.

We denote the solution set of Problem (5.1) by

$$\Delta = \Big\{x^* \in C \cap Fix(T_1) \text{ such that } Ax^* \in Q \cap Fix(T_2)\Big\}. \tag{5.2}$$

In sequel, we assume that $\Delta \neq \emptyset$.

5.2. **Preliminary Results.** The following well-known results are significant in proving the main result of this chapter.

Lemma 5.1. Let C be a nonempty closed convex subset of a Hilbert space.

(i) If G and S are two quasi-nonexpansive mappings on C, then GS is also quasi- nonexpansive mapping.

(ii) Let $G_\alpha = (1-\alpha)I + \alpha G$, where $\alpha \in (0,1]$ and G is a quasi-nonexpansive mapping on C. Then for all $x \in C$ and $q \in Fix(G)$, G_α is also a quasi-nonexpansive.

(iii) Let $\{G_i\}_{i=1}^N : C \to C$ be N-quasi-nonexpansive mappings and $\{\alpha_i\}_{i=1}^N$ be a positive sequence in (0,1) such that $\sum_{i=1}^N \alpha_i = 1$. Suppose that $\{G_i\}_{i=1}^N$ has a fixed point. Then

$$Fix\left(\sum_{i=1}^N \alpha_i G_i\right) = \bigcap_{i=1}^N Fix(G_i).$$

(iv) Let $\{G_i\}_{i=1}^N$ and $\{\alpha_i\}_{i=1}^N$ be as in (iii) above. Then $\sum_{i=1}^N \alpha_i G_i$, is a quasi-nonexpansive mapping. Furthermore, if for each $i = 1,2,3,...N$, $G_i - I$ is demiclosed at zero, then $\sum_{i=1}^N \alpha_i G_i - I$ is also.

The proof of (i) follows trivially, for the proof of (ii) see Moudafi [28] while the proof of (iii) and (iv) are deduce from Li and He [30].

6. Ishikawa-type Extra-Gradient Iterative Methods for Quasi-Nonexpansive Mappings in Hilbert Spaces

Theorem 6.1. Let $T : C \to H_1$ and $G : Q \to H_2$ be two quasi nonexpansive mappings and $A : H_1 \to H_2$ be a bounded linear operator with its adjoint A^*. Assume that $(T - I)$ and $(GP_Q - I)$ are demiclosed at zero, and $\Delta \neq \emptyset$. Define $\{x_n\}$ by

$$\begin{cases} x_0 \in C \;\; \text{chosen arbitrarily}, \\ y_n = P_C(x_n - \gamma_n A^*(I - GP_Q)Ax_n), \\ z_n = P_C(y_n - \gamma_n A^*(I - GP_Q)Ay_n), \\ w_n = (1-\alpha_n)z_n + \alpha_n T\Big((1-\beta_n)z_n + \beta_n T z_n\Big), \\ C_{n+1} = \Big\{z \in C_n : \|w_n - z\|^2 \leq \|z_n - z\|^2 \leq \|y_n - z\|^2 \leq \|x_n - z\|^2\Big\}, \\ x_{n+1} = P_{C_{n+1}}(x_0), \forall n \geq 0, \end{cases} \tag{6.1}$$

where P is a projection operator, $0 < a < \alpha_n < 1$, $0 < b < \beta_n < 1$, and $0 < c < \gamma_n < \frac{1}{L}$, with $L = \|AA^*\|$. Then $x_n \to x^* \in \Delta$.

Proof. **Step 1.** First, we show that $P_{C_{n+1}}$ is well defined. To show this, it suffices to show that for each $n \geq 0$, C_n is closed and convex. Trivially, C_n is closed.

Next, we show that C_n is convex. To show this, it suffices to show that for each $r_1, r_2 \in C_n$ and $\xi \in (0,1)$, $\xi r_1 + (1-\xi)r_2 \in C_n$.

Now, we compute

$$\begin{aligned} \|w_n - \xi r_1 - (1-\xi)r_2\|^2 &= \|\xi(w_n - r_1) + (1-\xi)(w_n - r_2)\|^2 \\ &= \xi\|w_n - r_1\|^2 + (1-\xi)\|w_n - r_2\|^2 - (1-\xi)\xi\|r_1 - r_2\|^2 \\ &\leq \xi\|z_n - r_1\|^2 + (1-\xi)\|z_n - r_2\|^2 - (1-\xi)\xi\|r_1 - r_2\|^2 \\ &= \|z_n - \xi r_1 - (1-\xi)r_2\|^2. \end{aligned} \tag{6.2}$$

Similarly, we obtain that

$$\begin{aligned} \|z_n - \xi r_1 - (1-\xi)r_2\|^2 &\leq \|y_n - \xi r_1 - (1-\xi)r_2\|^2 \\ &\leq \|x_n - \xi r_1 - (1-\xi)r_2\|^2. \end{aligned} \tag{6.3}$$

Thus, for each $r_1, r_2 \in C_n$, $\xi r_1 + (1-\xi)r_2 \in C_n$.

Step 2. Here, we show that $\Delta \subset C_n$, $n \geq 0$.

Let $q \in \Delta$ and $u_n = (1-\beta_n)z_n + \beta_n T z_n$. The fact that T is quasi-nonexpansive, it follows from (6.1) that

$$\begin{aligned}
\|w_n - q\|^2 &= \|(1-\alpha_n)z_n + \alpha_n T u_n - q\|^2 \\
&= \|(1-\alpha_n)(z_n - q) + \alpha_n(Tu_n - q)\|^2 \\
&= (1-\alpha_n)\|z_n - q\|^2 + \alpha_n \|Tu_n - q\|^2 - \alpha_n(1-\alpha_n)\|Tu_n - z_n\|^2 \\
&\leq (1-\alpha_n)\|z_n - q\|^2 + \alpha_n \|u_n - q\|^2 - \alpha_n(1-\alpha_n)\|Tu_n - z_n\|^2 \\
&\leq (1-\alpha_n)\|z_n - q\|^2 + \alpha_n \|(1-\beta_n)(z_n - q) + \beta_n(Tz_n - q)\|^2 \\
&- \alpha_n(1-\alpha_n)\|Tu_n - z_n\|^2 \\
&= (1-\alpha_n)\|z_n - q\|^2 + \alpha_n(1-\beta_n)\|z_n - q\|^2 + \alpha_n\beta_n\|Tz_n - q\|^2 \\
&- \alpha_n\beta_n(1-\beta_n)\|Tz_n - z_n\|^2 - \alpha_n(1-\alpha_n)\|Tu_n - z_n\|^2 \\
&\leq (1-\alpha_n)\|z_n - q\|^2 + \alpha_n(1-\beta_n)\|z_n - q\|^2 + \alpha_n\beta_n\|z_n - q\|^2 \\
&- \alpha_n\beta_n(1-\beta_n)\|Tz_n - z_n\|^2 - \alpha_n(1-\alpha_n)\|Tu_n - z_n\|^2 \\
&\leq \|z_n - q\|^2 . \qquad (6.4)
\end{aligned}$$

On the other hand, since G and P_C are both quasi-nonexpansive, by Lemma (5.1), we obtain that GP_C is also quasi-nonexpansive. Thus, we have

$$\begin{aligned}
\|z_n - q\|^2 &= \|P_C(y_n - \gamma_n A^*(I - GP_Q)Ay_n) - q\|^2 \\
&\leq \|y_n - \gamma_n A^*(I - GP_Q)Ay_n - q\|^2 \\
&= \|y_n - q\|^2 - 2\gamma_n \langle y_n - q, A^*(I - GP_Q)Ay_n\rangle + \|\gamma_n A^*(I - GP_Q)Ay_n\|^2 \\
&= \|y_n - q\|^2 - 2\gamma_n \langle Ay_n - GP_QAy_n + GP_QAy_n - Aq, Ay_n - GP_QAy_n\rangle \\
&+ \gamma_n^2 L\|(I - GP_Q)Ay_n\|^2 \\
&= \|y_n - q\|^2 + 2\gamma_n \langle Aq - GP_QAy_n, Ay_n - GP_QAy_n\rangle \\
&- \gamma_n(2 - \gamma_n L)\|GP_QAy_n - Ay_n\|^2 \\
&\leq \|y_n - q\|^2 - \gamma_n(1 - \gamma_n L)\|GP_QAy_n - Ay_n\|^2 . \qquad (6.5)
\end{aligned}$$

Following the same way as in the proof of (6.5), we obtain that

$$\|y_n - q\|^2 \leq \|x_n - q\|^2 . \qquad (6.6)$$

Combine Equation (6.4) $-$ (6.6), we have

$$\|w_n - q\|^2 \leq \|z_n - q\|^2 \leq \|y_n - q\|^2 \leq \|x_n - q\|^2. \qquad (6.7)$$

Thus, we have that $q \in C_n$, this implies that $\Delta \subset C_n$.

Noticing that $\Delta \subset C_{n+1} \subset C_n$ and $x_{n+1} = P_{C_{n+1}}(x_0) \subset C_n$, we have that

$$\|x_{n+1} - x_0\| \leq \|q - x_0\|, \forall n \geq 0 \text{ and } q \in \Delta. \qquad (6.8)$$

This shows that $\{x_n\}$ is bounded. By Lemma (2.43) we have

$$\begin{aligned}
\|x_{n+1} - x_n\|^2 + \|x_{n+1} - x_0\|^2 &= \|P_{C_{n+1}}(x_0) - x_n\|^2 + \|P_{C_{n+1}}(x_0) - x_0\|^2 \\
&\leq \|x_n - x_0\|^2. \qquad (6.9)
\end{aligned}$$

This implies that

$$\|x_{n+1} - x_0\| \leq \|x_n - x_0\|.$$

Thus, $\{\|x_n - x_0\|\}$ is a non-increasing sequence and bounded below by zero. Therefore, $\lim_{n\to\infty} \|x_n - x_0\|$ exists.

On the other hand, for each $k > n$, we also obtain that

$$\begin{aligned} \|x_k - x_n\|^2 + \|x_n - x_0\|^2 &= \|P_{C_n}(x_0) - x_k\|^2 + \|P_{C_n}(x_0) - x_0\|^2 \\ &\leq \|x_k - x_0\|^2. \end{aligned} \tag{6.10}$$

Thus, by (6.10) and the fact that the $\lim_{n\to\infty} \|x_{n+1} - x_0\|$ exist, we obtain that

$$\lim_{k,n\to\infty} \|x_k - x_n\| = 0.$$

This shows that $\{x_n\}$ is Cauchy.

Since $x_{n+1} = P_{C_{n+1}}(x_0) \in C_{n+1} \subset C_n$ and the fact that $\{x_n\}$ is a Cauchy sequence, we deduce that

$$\begin{aligned} \|y_n - x_n\| &\leq \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \\ &\leq 2\|x_{n+1} - x_n\|, \end{aligned}$$

and

$$\begin{aligned} \|z_n - x_n\| &\leq \|z_n - x_{n+1}\| + \|x_{n+1} - x_n\| \\ &\leq 2\|x_{n+1} - x_n\|. \end{aligned}$$

Thus, as $n \to \infty$, we deduce that

$$\lim_{n\to\infty} \|z_n - x_n\| = 0 \text{ and } \lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{6.11}$$

On the other hand,

$$\|y_n - z_n\| \leq \|y_n - x_n\| + \|x_n - z_n\|.$$

By (6.11) we obtain that

$$\lim_{n\to\infty} \|y_n - z_n\| = 0. \tag{6.12}$$

Similarly, we obtain that

$$\lim_{n\to\infty} \|w_n - x_n\| = 0, \lim_{n\to\infty} \|z_n - y_n\| = 0 \text{ and } \lim_{n\to\infty} \|w_n - z_n\| = 0. \tag{6.13}$$

By (6.5), we obtain that

$$\begin{aligned} \|GP_Q Ay_n - Ay_n\|^2 &\leq \frac{\|y_n - x^*\|^2 - \|z_n - x^*\|^2}{\gamma_n(1 - \gamma_n L)} \\ &\leq \frac{\|y_n - z_n\| \Big(\|z_n - y_n\| + 2\,\|z_n - x^*\| \Big)}{c(1 - \gamma_n L)}. \end{aligned}$$

Thus, by (6.12) we deduce that

$$\lim_{n\to\infty} \|GP_Q Ay_n - Ay_n\| = 0.$$

Similarly, by (6.4) and (6.13), we deduce that

$$\lim_{n\to\infty} \|Tz_n - z_n\| = 0.$$

Finally, we show that $x_n \to x^*$.

Since $\{x_n\}$ is Cauchy, we assume that $x_n \to p$. By Equation (6.1), we have that $z_n \to p$, this implies that $z_n \rightharpoonup p$. The fact that the $\lim_{n\to\infty} \|Tz_n - z_n\| = 0$ together with the demiclosed of $(T - I)$ at zero, we deduce that $p \in Fix(T)$.

On the other hand, since $x_n \to p$, implies that $y_n \rightharpoonup p$, by (6.1), we deduce that $p = P_C p$ which implies that $p \in C$ and therefore we have $p \in C \cap Fix(T)$.

Furthermore, by the definition of A, we have that $Ay_n \to Ap$, this implies that $Ay_n \rightharpoonup Ap$. The fact that the $\lim_{n\to\infty} \|GP_Q Ay_n - Ay_n\| = 0$ together with the demiclosed of $(GP_Q - I)$ at zero, we deduce that $Ap \in Fix(GP_Q)$, this implies that $Ap \in Q \cap Fix(G)$. Hence $p \in \Delta$. This show that $x_n \to x^*$. The proof is complete. $\square$

Next, we consider the split feasibility and fixed point problems for the class of finite family of quasi-nonexpansive mappings.

Theorem 6.2. Let $\{T_i\}_{i=1}^M : C \to H_1$ and $\{G_j\}_{j=1}^N : Q \to H_2$ be quasi nonexpansive mappings with $\bigcap_{i=1}^M Fix(T_i) \neq \emptyset$ and $\bigcap_{j=1}^N Fix(G_j) \neq \emptyset$. And also let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint A^*. Assume that $(T_i - I), i = 1, 2, 3, ..., M$ and $(G_j P_Q - I), j = 1, 2, 3, ..., N$ are demiclosed at zero, and $\Delta \neq \emptyset$. Define $\{x_n\}$ by

(6.14)
$$\begin{cases} x_0 \in C \ \text{ chosen arbitrarily}, \\ y_n = P_C(x_n - \gamma_n A^*(I - \sum_{j=1}^N \delta_j G_j P_Q)Ax_n), \\ z_n = P_C(y_n - \gamma_n A^*(I - \sum_{j=1}^N \delta_j G_j P_Q)Ay_n), \\ w_n = (1-\alpha_n)z_n + \alpha_n \sum_{i=1}^M \lambda_i T_i\Big((1-\beta_n)z_n + \beta_n \sum_{i=1}^M \lambda_i T_i z_n\Big), \\ C_{n+1} = \Big\{z \in C_n : \|w_n - z\|^2 \leq \|z_n - z\|^2 \leq \|y_n - z\|^2 \leq \|x_n - z\|^2\Big\}, \\ x_{n+1} = P_{C_{n+1}}(x_0), \forall n \geq 0, \end{cases}$$

where P is a projection operator, $0 < a < \alpha_n < 1$, $0 < b < \beta_n < 1$, and $0 < c < \gamma_n < \frac{1}{L}$ with $L = \|AA^*\|$. Then $x_n \to x^* \in \Delta$.

Proof. □

By Lemma 5.1, we deduce that

(i) $\sum_{j=1}^N \delta_j G_j$ and $\sum_{i=1}^M \lambda_i T_i$ are quasi-nonexpansive mappings.
(ii) $\sum_{j=1}^N \delta_j (G_j - I)$ and $\sum_{i=1}^M \lambda_i (T_i - I)$ are demiclosed at zero.
(iii) $Fix\left(\sum_{i=1}^M \lambda_i T_i\right) = \bigcap_{i=1}^M Fix(T_i)$ and $Fix\left(\sum_{j=1}^N \delta_j G_j\right) = \bigcap_{j=1}^N Fix(G_j)$.

Thus, all the hypothesis of Theorem 6.2 is satisfied. Therefore, the proof of this theorem follows trivially from Theorem 6.1.

As the consequence of Theorem 6.2, we immediately obtain the following corollary.

Corollary 6.3. Let $\{T_i\}_{i=1}^M : C \to H_1$ and $\{G_j\}_{j=1}^N : Q \to H_2$ be quasi nonexpansive mappings with $\bigcap_{i=1}^M Fix(T_i) \neq \emptyset$ and $\bigcap_{j=1}^N Fix(G_j) \neq \emptyset$, respectively. And also let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint A^*. Assume that $(T_i - I), i = 1, 2, 3, ..., M$ and $(G_j P_Q - I), j = 1, 2, 3, ..., N$ are demiclosed at zero and $\Delta \neq \emptyset$. Define $\{x_n\}$ by

$$\begin{cases} x_0 \in C \ \text{ chosen arbitrarily}, \\ z_n = P_C(x_n - \gamma_n A^*(I - \sum_{j=1}^N \delta_j G_j P_Q)Ax_n), \\ w_n = (1-\alpha_n)z_n + \alpha_n \sum_{i=1}^M \lambda_i T_i z_n, \\ C_{n+1} = \Big\{z \in C_n : \|w_n - z\|^2 \leq \|z_n - z\|^2 \leq \|x_n - z\|^2\Big\}, \\ x_{n+1} = P_{C_{n+1}}(x_0), \forall n \geq 0, \end{cases} \tag{6.15}$$

where P is a projection operator, $0 < a < \alpha_n < 1$, $0 < b < \beta_n < 1$, and $0 < c < \gamma_n < \frac{1}{L}$ with $L = \|AA^*\|$. Then $x_n \to x^* \in \Delta$.

Proof. In Algorithm (6.14), take $y_n = x_n$ and $\beta_n = 0$, then, Algorithm (6.14) reduces to Algorithm (6.15); therefore, all the hypothesis of Theorem 6.2 is satisfied. Hence, the proof of this corollary follows directly from Theorem 6.2. □

6.1. **Application to Split Feasibility Problems.** As a special case of Problem (5.2), we give the following theorems for solving split feasibility Problem and the fixed point problem.

Theorem 6.4. Let $\{T_i\}_{i=1}^M : C \to H_1$ and $\{G_j\}_{j=1}^N : Q \to H_2$ be quasi-nonexpansive mappings with $\bigcap_{i=1}^M Fix(T_i) \neq \emptyset$ and $\bigcap_{j=1}^N Fix(G_j) \neq \emptyset$. And also let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint A^*. Assume that $(T_i - I), i = 1, 2, 3, ..., M$ and $(G_j - I), j = 1, 2, 3, ..., N$ are demiclosed at zero and $\Delta \neq \emptyset$. Define $\{x_n\}$ by

(6.16)
$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily,} \\ y_n = P_C(x_n - \gamma_n A^*(I - \sum_{j=1}^N \delta_j G_j)Ax_n), \\ z_n = P_C(y_n - \gamma_n A^*(I - \sum_{j=1}^N \delta_j G_j)Ay_n), \\ w_n = (1-\alpha_n)z_n + \alpha_n \sum_{i=1}^M \lambda_i T_i\Big((1-\beta_n)z_n + \beta_n \sum_{i=1}^M \lambda_i T_i z_n\Big), \\ C_{n+1} = \Big\{z \in C_n : \|w_n - z\|^2 \leq \|z_n - z\|^2 \leq \|y_n - z\|^2 \leq \|x_n - z\|^2\Big\}, \\ x_{n+1} = P_{C_{n+1}}(x_0), \forall n \geq 0, \end{cases}$$

where P is a projection operator, $0 < a < \alpha_n < 1$, $0 < b < \beta_n < 1$, and $0 < c < \gamma_n < \frac{1}{L}$ with $L = \|AA^*\|$. Then $x_n \to x^* \in \Omega$.

Proof. In Algorithm (6.14), take $P_Q = I$ (the identity mapping), then, Algorithm (6.14) reduces to Algorithm (6.16), therefore, all the hypothesis of Theorem 6.2 is satisfied. Hence, the proof of this theorem, follows directly from Theorem 6.2. □

Theorem 6.5. Let $\{T_i\}_{i=1}^M : C \to H_1$ be quasi-nonexpansive mapping with $\bigcap_{i=1}^M Fix(T_i) \neq \emptyset$. Assume that $(T_i - I), i = 1, 2, 3, ..., M$ is demiclosed at zero. Define $\{x_n\}$ by

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily,} \\ u_n = (1-\beta_n)x_n + \beta_n \sum_{i=1}^M \lambda_i T_i x_n, \\ w_n = (1-\alpha_n)u_n + \alpha_n \sum_{i=1}^M \lambda_i T_i u_n, \\ C_{n+1} = \Big\{z \in C_n : \|w_n - z\|^2 \leq \|u_n - z\|^2 \leq \|x_n - z\|^2\Big\}, \\ x_{n+1} = P_{C_{n+1}}(x_0), \forall n \geq 0, \end{cases} \tag{6.17}$$

where P is a projection operator, $0 < a < \alpha_n < 1$ and $0 < b < \beta_n < 1$. Then $\{x_n\}$ converges strongly to the solution of common fixed point of $\{T_i\}_{i=1}^M$.

Proof. In Algorithm (6.14), take $\gamma_n = 0$ and $P_C = I$ (the identity mapping), then, Algorithm (6.14) reduces to Algorithm (6.17), therefore; all the hypothesis of Theorem 6.2 is satisfied. Hence, the proof of this theorem, follows directly from Theorem 6.2. □

6.1.1. *Conclusion.* In this section, we have proposed Ishikawa-type extra-gradient methods for solving the split feasibility and fixed point problems for the class of quasi-nonexpansive mappings in Hilbert spaces. Under some suitable assumptions imposed on some parameters and operators involved, we proved the strong convergence theorems of these algorithms. Furthermore, as an application, we gave the strong convergence theorem for the split feasibility problem. The results presented in this chapter, not only extend the result of Chen et al., [77] but also extend, improve and generalize the results of; Takahashi and Toyoda [78], Nadezhkina and Takahashi [79], Ceng et al., [80] and Li and He [30] in the following ways:

- The theorem of Chen et al., [77] gave the weak convergence results while ours gave the strong convergence results.
- The technique of proving our results is entirely different from that of Chen et al., [77]. Furthermore, the algorithms of Chen et al., [77] involve the class of nonexpansive mappings while our algorithm includes the class of quasi-nonexpansive mappings which are more general that nonexpansive mappings.
- The method for finding the solution of the split feasibility and fixed point problems is more general that the method of finding the solution to split feasibility problem.
- The theorem of Li and He [30] gave the strong convergence results for the split feasibility problem while ours gave the strong convergence for the split feasibility and fixed point problems. Furthermore, our algorithms generalize that of Li and He [30]. For instance, in Theorem 6.1 Algorithm (6.1) take $y_n = x_n$ and $\beta_n = 0$, hence, our algorithm reduces to that of Li and He [30] Theorem 2.1 Algorithm 2.1.
- Our theorems gave the strong convergence for the solution of the split feasibility and fixed point problems for the class of quasi-nonexpansive mappings, while the results of Ceng et al., [39] gave a weak convergence result for the solution of the split feasibility and fixed point problems for the class of nonexpansive mappings.
- The split feasibility and fixed point problems is a fascinating problem. It generalizes the split feasibility problem (SFP) and fixed point problem (FPP). All the results and conclusions that are true for the split feasibility and fixed point problems continue to holds for these problems (SFP, FPP), and it shows the significance and the range of applicability of split feasibility and fixed point problems.
- The novelty of our theorems gives strong convergence results while the theorem of; [78] Nadezhkina and Takahashi [79], Ceng et al., [80] and Li and He [30] all gives weak convergence results.

7. Conclusion

In this work, we have studied the split common fixed point problems and its applications. We have suggested some algorithms for solving this split common fixed point problems and its variant forms for different classes of nonlinear mappings.

Proceeding systematically in our work, we gave the basic definitions and results from the literature. Also, we briefly provided an overview of the split common fixed point problems and its variant forms in Section 2. In the next section, we have suggested and analysed iterative algorithms for solving the split common fixed point problems for the class of total quasi asymptotically nonexpansive mappings in Hilbert spaces. Also, we gave the strong convergence results of the proposed algorithms. Also, we considered an algorithm for solving this split common fixed point problems for the class of demicontractive mappings without any prior information on the normed on the bounded linear operator and established the strong convergence results of the proposed algorithm.

As a generalization of the split feasibility problem, we proposed Ishikawa-type extra-gradient methods for solving the split feasibility and fixed point problems for the class of quasi-nonexpansive mappings in Hilbert spaces. Under some suitable assumptions imposed on some parameters and operators involved, we proved the strong convergence theorems of these algorithms.

In the end, we proposed a new problem called "Split Feasibility and Fixed Point Equality Problems (SFFPEP)" and study it for the class of quasi-nonexpansive mappings in Hilbert spaces. We also proposed new iterative methods for solving this SFFPEP and proved the convergence results of the proposed algorithms. In additions, as a generalization of SFFPEP, we consider another problem called "Split Common Fixed Equality Problems (SCFPEP)" and study it for the class of finite family of quasi-nonexpansive mappings in Hilbert spaces. Finally, We suggested some algorithms for solving this SCFPEP and proved the convergence results of the proposed algorithms.

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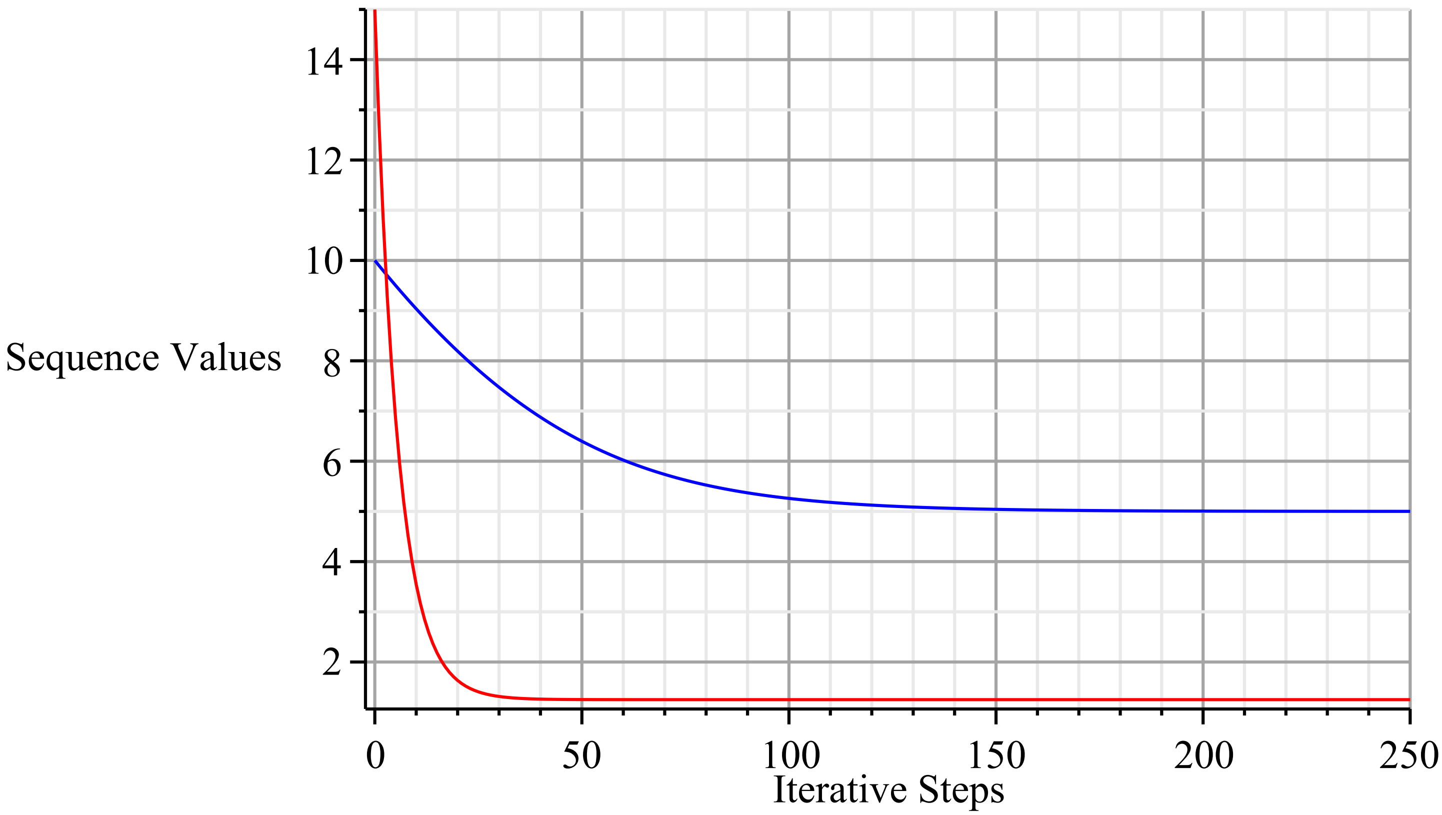

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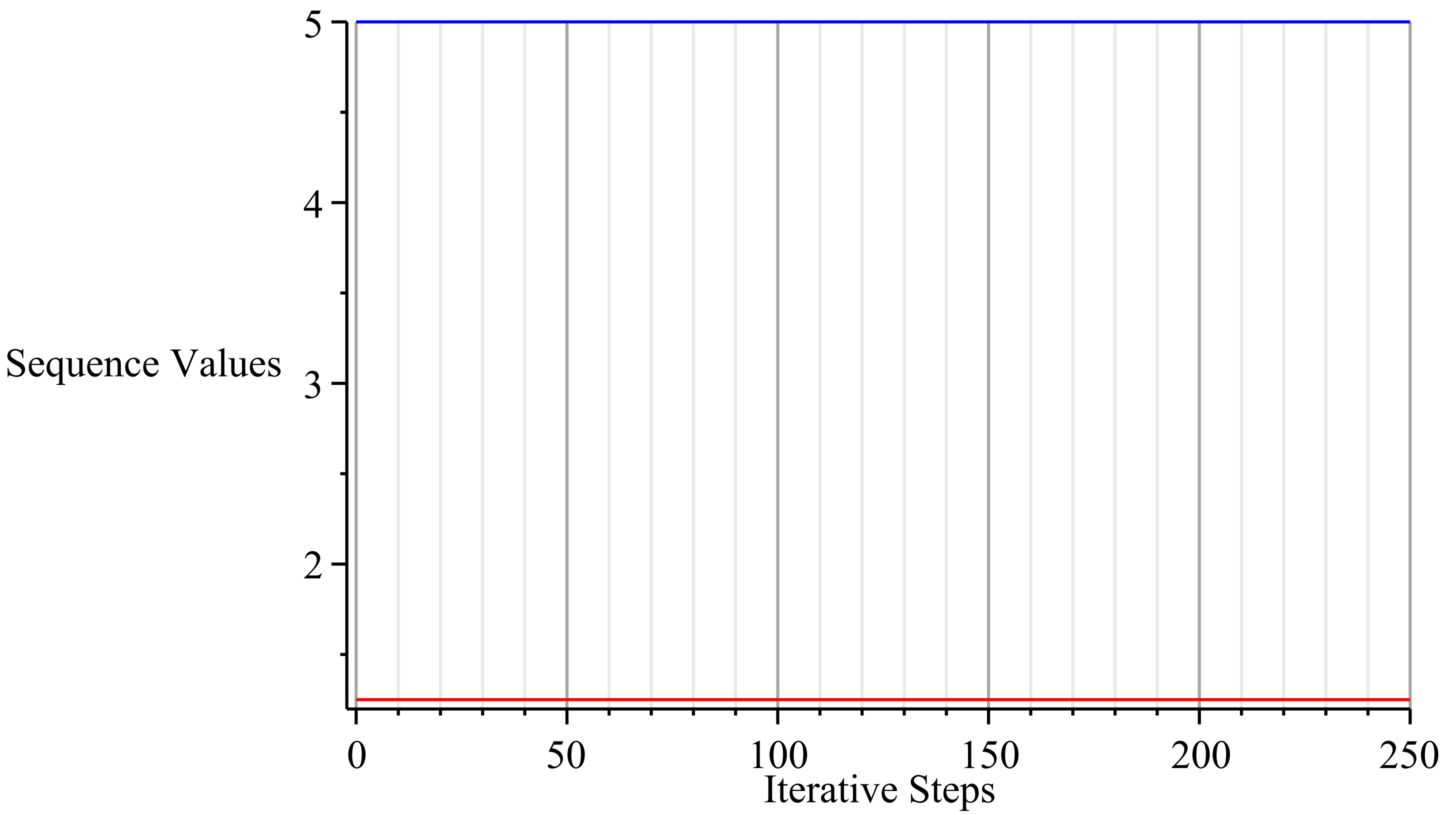
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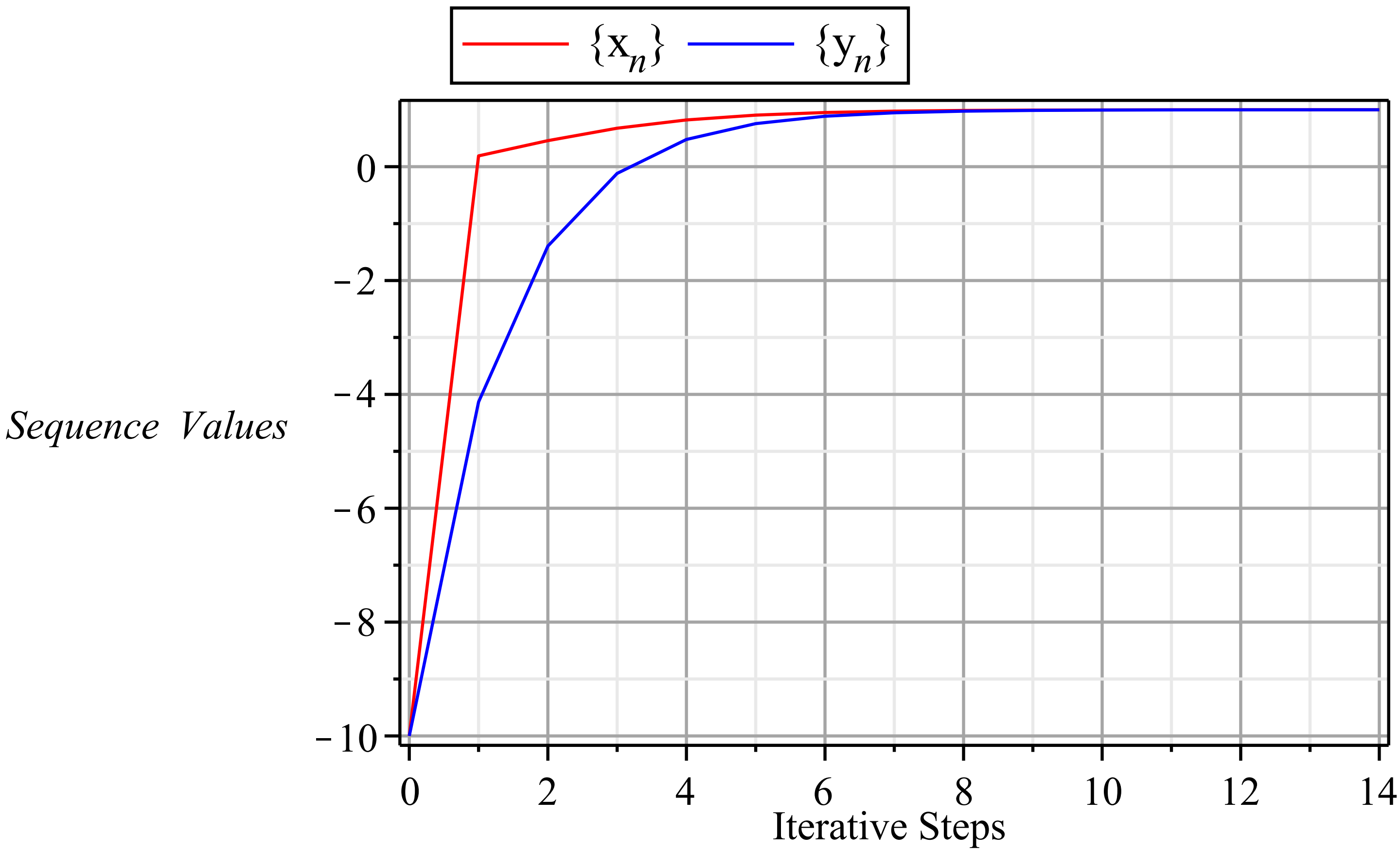

{x_n}
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−6
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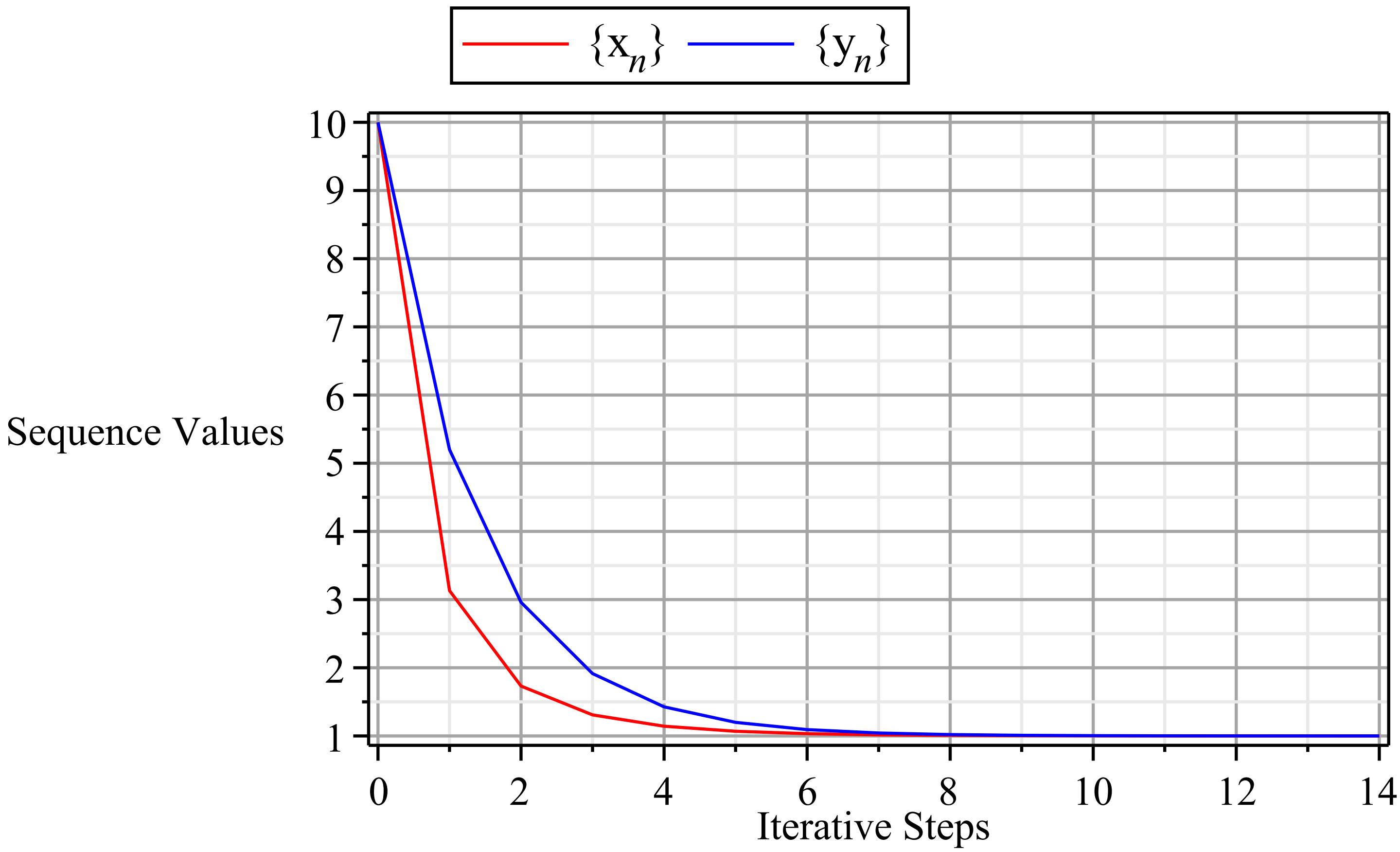
12
14
Iterative Steps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{y_n}
Sequence Values
Iterative Steps
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2
3
4
5
6
7
8
9
10
0
2
4
6
8
10
12
14

```
M Digits a 10 :
M with(plots) :
M x[0] a I 5 : y[0] a I 5 :
M for n from 0 by 1 to 149 do
   k[n] a n;
   z[n] a evalf(x[n]);
   w[n] a evalf( (8 z[n])/9 A 1/9 ( (2 z[n])/3 A (2 z[n])/(3(z[n] A 1)) ) );
   x[n A 1] a evalf( (4 z[n])/5 A 1/5 ( (2 w[n])/3 A (2 w[n])/(3(w[n] A 1)) ) );

   u[n] a evalf(y[n]);
   r[n] a evalf( (8 u[n])/9 A 1/9 ( (4 u[n])/5 A (u[n] A 2)/15 ) );
   y[n A 1] a evalf( (6 u[n])/7 A 1/7 ( (4 r[n])/5 A (r[n] A 2)/15 ) );
   end do;
M Y a [seq(y[i], i = 0..149)] :
M X a [seq(x[i], i = 0..149)] :
M N a [seq(k[i], i = 0..149)] :
M P1 a plot(N, X) :
M P2 a plot(N, Y) :
M plots[display](P1, P2) :
```